Minimizing Backlash in Spur Gears

Richard L. Thoen
Consultant,
Minneapolis, MN

Nomenclature

- B — Arc on both reference circles that corresponds to B_b on both base circles
- B_b — Backlash on line of action
- C — Distance between centers of reference circles
- C_b — Basic center distance, $(N + n)/2P$ or $m(N + n)/2$
- m — Module
- N — Number of teeth on gear
- n — Number of teeth on pinion
- P — Diametral Pitch
- p — Circular pitch on both reference circles, π/P or πm
- $p/2$ — basic tooth thickness
- R — Radius of gear reference circle, $N/2P$ or $mN/2$
- r — Radius of pinion reference circle, $n/2P$ or $mn/2$
- R_b — Radius of gear base circle
- r_b — Radius of pinion base circle
- S — Arc on both reference circles that corresponds to S_b on both base circles
- S_b — Space on line of action due to an increase in center distance
- ΔT — Deviation from $p/2$ on gear reference circle
- Δt — Deviation from $p/2$ on pinion reference circle
- ΔT_b — Change in tooth thickness on gear base circle
- Δt_b — Change in tooth thickness on pinion base circle
- Φ — Profile angle of generating rack
- ϕ — Pressure angle

Abstract

Simplified equations for backlash and roll test center distance are derived. Unknown errors in measured tooth thickness are investigated. Master gear design is outlined, and an alternative to the master gear method is described. Defects in the test radius method are enumerated. Procedures for calculating backlash and for preventing significant errors in measurement are presented.

Introduction

An important part of designing for minimum backlash is the prevention of significant errors in measurement. Circular tooth thickness, for instance, is not measured directly, but is calculated from an equation that presumes perfect teeth. As a result, a master gear — even if perfect — may not indicate the center distance at which imperfect gears will mesh tightly with each other. Moreover, the magnitude of the error in measured tooth thickness varies with the number of teeth on the master gear.

Measured tooth thickness also varies with depth of contact and active face width, so the master gear should be representative of the gear that mates with the work gear. But most measurements are made with general purpose, off-the-shelf master gears.

Significant errors exist in many, if not most, roll test measurements. Evidence for this is the discrepancies in measurement that arise when the same gear is inspected by different people (buyer and seller, inspection and production departments, different inspectors using the same test equipment, etc.). The typical response has been to standardize the test equipment and inspection procedure — a maneuver which can reduce the discrepancies in measurement, but not necessarily the errors in measurement. The purpose of this article is to show how to deal with errors in measurement when designing for minimum backlash.

Basic Geometry

A very simple way of treating backlash is to start with no backlash (Fig. 1) and then examine the effect of an increase in center distance (Fig. 2).

In Fig. 1 the terms *basic center distance* (C_b), *profile angle* (Φ) and *reference circle* — which have been around for at least 30 years (Refs. 1–2) — are used in lieu of *standard center distance*, *standard pressure angle* and *standard pitch circle*. Consequently, in Fig. 2 there is no need for the qualifying adjective *operating*, which means that *operating center distance*, *operating pressure angle* and *operating pitch circle* can be shortened to *center distance* (C), *pressure angle* (ϕ) and *pitch circle*.

From Fig. 1 it is seen that

$$x + y = R_b \tan \Phi + r_b \tan \Phi,$$

and from Fig. 2 that

$$R_b(\phi - \Phi) + x + \frac{S_b}{2} + y + r_b(\phi - \Phi)$$

$$= R_b \tan \phi + r_b \tan \phi,$$

which, upon substitution for $x + y$, becomes

$$\frac{S_b}{2} + R_b(\tan \Phi - \Phi) + r_b(\tan \Phi - \Phi) =$$

$$R_b(\tan \phi - \phi) + r_b(\tan \phi - \phi),$$

where $\tan \Phi - \Phi = \text{inv } \Phi$ and $\tan \phi - \phi = \text{inv } \phi$, so that

$$S_b = 2(R_b + r_b)(\text{inv } \phi - \text{inv } \Phi). \quad (1)$$

This equation was first derived in a somewhat different way by Candee (Ref. 3), who remarked that it is "the shortest and most direct equation," and "does not require the determination of tooth thickness on new pitch circles" — an apparent reference to the well-known derivation by Buckingham (Ref. 4).

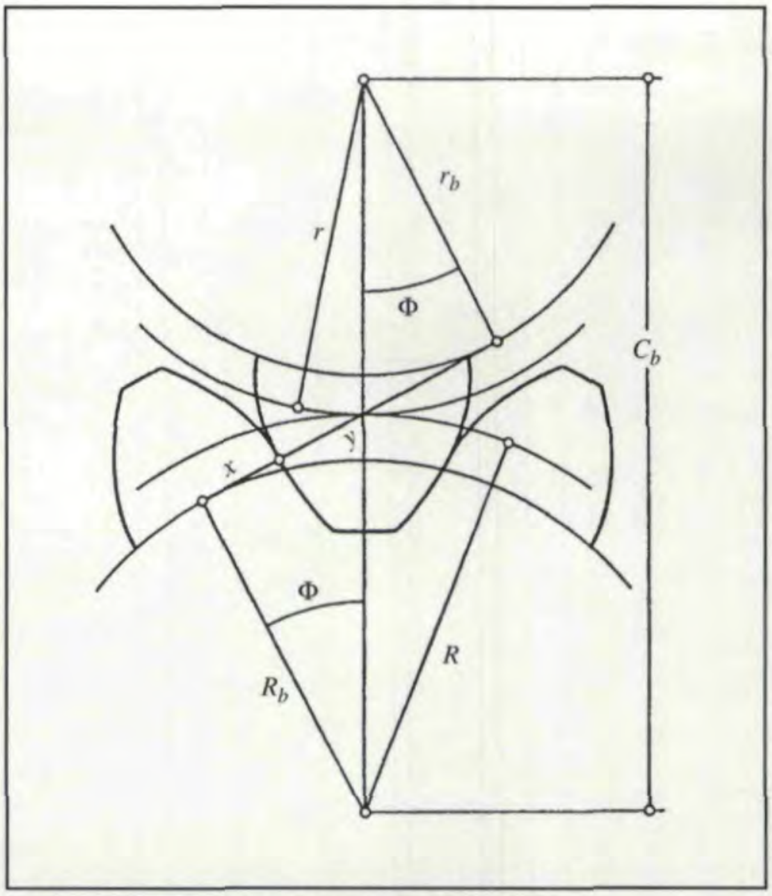

Fig. 1 — Reference circles and tooth profiles in contact.

From Fig. 2 and from the definition of the involute (the curve generated by a point on a string as it is being unwound from a circle), it is seen that $S_b/2$ can be wrapped onto either base circle. Consequently, with the gear fixed, the pinion is free to rotate $S_b/2r_b$ radians, both *CW* and *CCW*, or S_b/r_b total. Likewise, with the pinion fixed, the gear is free to rotate $S_b/2R_b$ radians, both CW and CCW, or S_b/R_b total.

It should be observed that space S_b could be eliminated by increasing the tooth thickness on either the gear (by setting $\Delta T_b = S_b$), or the pinion ($\Delta t_b = S_b$) or some combination of both ($\Delta T_b + \Delta t_b = S_b$). In general, with an allowance for backlash,

$$S_b = \Delta T_b + \Delta t_b + B_b, \quad (2)$$

where S_b is obtained from Eq. 1. And it should be observed that if the teeth were thinned, then both ΔT_b and Δt_b would be negative, which would make B_b greater than S_b.

Richard L. Thoen

is a consultant specializing in medium and fine-pitch gearing. He is the author of several articles and papers on measurement, involute mathematics, statistical tolerancing and other gearing subjects.

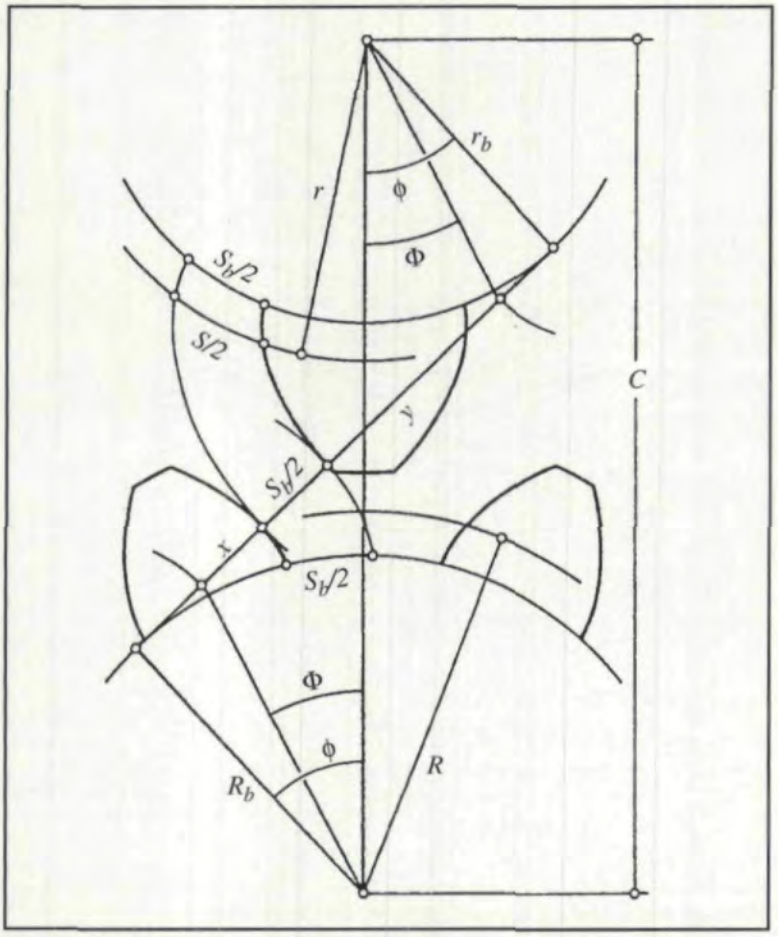

Fig. 2 — Effect of an increase in center distance.

Since tooth thickness on the base circle varies with tooth number, it is customary to work with tooth thickness on the reference circle, where for all tooth numbers, the tooth thickness is, in most cases, nearly equal to the basic tooth thickness ($p/2$). Further, it is convenient to work with the deviations from $p/2$, namely, with ΔT and Δt, mainly because they can be exchanged for backlash (as will be seen in Eq. 4). These deviations from $p/2$ are, of course, related to the offset of the generating rack; i.e., $\Delta t = 2\Delta C \tan \Phi$ where ΔC is the rack offset.

The relation of an arc on the reference circle to an arc on the base circle is shown in Fig. 2, where it is seen that arcs $S_b/2$ and $S/2$ subtend equal angles. Specifically, for the pinion,

$$\frac{S_b/2}{r_b} = \frac{S/2}{r}, \text{ or } S_b = \frac{S_b}{r}$$

where, from Fig. 1, the $r_b = r \cos \Phi$, so that $S_b = S \cos \Phi$. Similarly, for Δt_b and B_b in Eq. 2, $\Delta t_b = \Delta t \cos \Phi$ and $B_b = B \cos \Phi$. Likewise, for the gear,

$$\frac{S_b/2}{R_b} = \frac{S/2}{R}, \text{ or } S_b = S\,\frac{S_b}{R}$$

where, from Fig. 1, the $R_b = R \cos \Phi$, so that $S_b = S \cos \Phi$, $\Delta T_b = \Delta T \cos \Phi$ *and* $B_b = B \cos \Phi$. Thus, from Eqs. 1 and 2 it is seen that

$$S_b = S \cos \Phi = (\Delta T + \Delta t + B) \cos \Phi = 2(R_b + r_b)(\text{inv } \phi - \text{inv } \Phi),$$

or

$$B = 2\,\frac{R_b + r_b}{\cos \Phi}\,(\text{inv } \phi - \text{inv } \Phi) - (\Delta T + \Delta t),$$

where, from Figs. 1 and 2, the

$$R_b + r_b = C_b \cos \Phi = C \cos \phi,$$

so that

$$\cos \phi = \frac{C_b}{C} \cos \Phi, \tag{3}$$

$$B = 2\,C_b(\text{inv } \phi - \text{inv } \Phi) - (\Delta T + \Delta t), \tag{4}$$

and

$$B_b = B \cos \Phi. \tag{5}$$

Unknown Errors

It is important to remember that ΔT and Δt (in Eq. 4) are subject to unknown errors in measured tooth thickness, because conventional measuring methods (span, pins, master gear) presume perfect teeth. For example, given a perfect $36T$-$12P$-20° gear the measured tooth thickness can be found from

$$M = \frac{10.8367}{12} + \Delta t \cos 20°,$$

where M is the span measurement across 4 teeth and 10.8367 is the span dimension for $P = 1$ and $\Delta t = 0$. Thus, for a span measurement of 10.8367/12, from

$$M = \frac{10.8367}{12} = \frac{10.8367}{12} + \Delta t \cos 20°,$$

the $\Delta t = 0$, and for two such gears meshed at a center distance of $C = C_b = (36 + 36)/(2 \times 12) = 3$ inches, the backlash along the line of action is zero; i.e., from Eqs. 3, 4, and 5, the $B_b = 0$. However, for the same span measurement on a gear that has been photographically reduced to $P = 12.05$, from

$$M = \frac{10.8367}{12} = \frac{10.8367}{12.05} + \Delta t \cos 20°,$$

the $\Delta t = 0.00399$, $C_b = 36/12.05$, and at $C = 3$, the $B_b = 0.0012$, not $B_b = 0$ as for $P = 12$. Similarly, for a gear that has been photographically enlarged to $P = 11.95$, the $\Delta t = -0.00402$, $C_b = 36/11.95$, and at $C = 3$ the $B_b = -0.0009$ (and interference).

For the aforesaid $36T$-$12P$-20° gear, the dimension over 1.92 pins is 39.0886 for $P = 1$ and $\Delta t = 0$. Thus, for a pin measurement of 39.0886/12 on a $P = 12.05$ gear, the $\Delta t = 0.00502$ (obtained by solving the pin equations for Δt), and for two such gears meshed at $C = 3$, the $B_b = -0.0008$ (an interference, versus a backlash of 0.0012 for span). Similarly, for the same pin measurement on a $P = 11.95$ gear, the $\Delta t = -0.00494$ and $B_b = 0.0008$ (versus an interference of 0.0009 for span). In short, for both $P = 12.05$ and 11.95, the backlash for pins was opposite in sign to the backlash for span.

It is of interest to note that the error in base pitch, relative to $P = 12$, was 0.0010 inches for both $P = 12.05$ and 11.95, and that an error of this magnitude is not uncommon in formed gearing (molded plastic, die cast, powder metal, stamped, cold-drawn). Further, when the combination of tolerances (for outside radius, tip round, bearing clearance, center distance) is large relative to whole depth, as in the case of fine-pitch formed gearing, it usually is necessary to design for minimum backlash, so as to avoid a contact ratio of less than unity (Ref. 5).

And it should be noted that no generalization can be drawn from these idealized examples, since the error in measured tooth thickness varies with the number of teeth spanned, the pin diameter and the tooth number.

When the two-flank roll test is not practical, the ΔT and Δt (in Eq. 4) should be adjusted to account for the allowable deviations in runout, tooth alignment, profile and spacing. An approximate adjustment can be deduced from similar designs, namely, from the discrepancy between calculated backlash and measured backlash — which also includes the aforementioned unknown error in measured tooth thickness. But if similar designs are not available, then it is necessary to estimate the individual adjustments (Ref. 6), and to assume a value for the unknown error in measured tooth thickness.

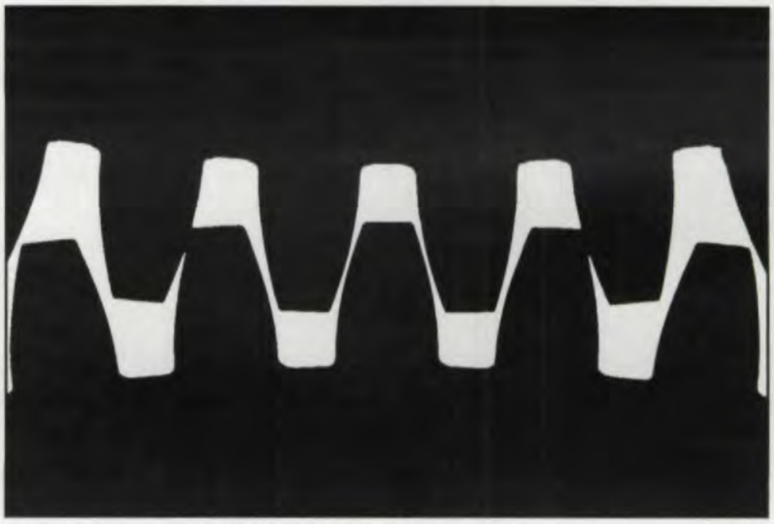

Fig. 3 — A mesh in which some teeth fail to make contact.

Characteristics of Master Gears

When a master gear is used to measure tooth thickness, the teeth will be thinner than the value indicated by the master gear, not thinner or thicker as when span or pins are used. To illustrate, consider two gears — one imperfect and one perfect — in mesh with a master gear at the same center distance. It is clear that the teeth on the imperfect gear will be thinner than the teeth on the perfect gear (Ref. 7). As in all functional gaging (gears, splines, screw threads, bores), size is sacrificed for defects in form and position. Consequently, since the teeth are thinner than that indicated by the master gear, the backlash will be somewhat greater than that indicated by Eq. 4.

Moreover, the magnitude of the error in measured tooth thickness will vary with the number of teeth on the master gear. This can be easily confirmed by meshing gears of various tooth numbers with a single gear of slightly different pitch and measuring both the tooth-to-tooth composite variation (also known as tooth-to-tooth composite tolerance, tooth-to-tooth composite error and originally as *kickout* [Refs. 8–9]) and center distance.

For example, when 192*T*-127*P*-20° and 180*T*-120*P*-20° master gears are rolled together, the kickout is only about 0.0002 inches, despite the fact that the difference in base pitch is 0.0014 inches. But when 120*P*-20° relatively high-grade gears (error in base pitch ≤10% of 0.0014 inches) of various tooth numbers are rolled with the 192*T*-127*P* master gear, the kickout increases as the tooth number decreases, to about 0.0019 inches for a 12*T*-120*P* pinion. This effect, which is well-known (Ref. 10), is the result of partial tooth contact (see Fig. 3), the degree of which varies with contact ratio. Conversely, the magnitude of the center distance error — namely, the amount by which the maximum center distance exceeds the sum of the reference radii — decreases as the tooth number decreases, from about 0.006 inches for the 180*T*-120*P* master gear to about 0.003 inches for a 12*T*-120*P* pinion.

A single master gear, therefore, should not be used to check any and all tooth numbers. The master gear method has, in Mark Twain's words, "a certain degree of merit . . ., but like chastity — it can be carried too far!"

Master Gear Design

The ideal master gear would have the same tooth number, same depth of contact and same pitch circle as the gear that mates with the gear to be inspected; i.e., with the work gear. And when the face width of the mating gear is less than the face width of the work gear, the ideal master gear would have the same face width as the active face width of the mating gear.

Because depth of contact depends upon the outside radius of the mating gear, the tip round or chamfer on the mating gear, and the center distance between mating gear and work gear, all three of which vary, the master gear should be designed to match the maximum depth of contact. Specifically, the outside radius of the master gear should be the same as the maximum outside form radius of the mating gear (maximum outside radius less the radial effect of minimum tip round or chamfer). And the tooth thickness of the master gear should be such that the maximum roll test center distance between master gear and work gear is the same as the minimum center distance between the mating gear and the work gear. That is, for $B = 0$ in Eq. 4, the

$$\Delta T = 2\, C_b(\mathrm{inv}\ \phi - \mathrm{inv}\ \Phi) - \Delta t, \qquad (6)$$

where ΔT is for the master gear, inv ϕ is obtained from Eq. 3 (wherein C is the minimum center distance between mating gear and work gear), and Δt is for maximum tooth thickness of the work gear.

The tooth thickness of most work gears will be less than maximum, so in most roll tests the depth of contact between master gear and work gear will be slightly greater than the maximum depth of contact between mating gear and work gear.

In most cases an acceptable approximation to the ideal tooth number can be obtained by simulating the roll test on a computer. In particular, the ideal master gear is meshed with the work gear, and then a kickout, equal to the kickout tolerance, is induced by altering the diametral pitch of the work gear while maintaining the center distance by reducing the tooth thickness of the work gear. Next, the tooth number is altered, but only to the point where there is no significant change in kickout or tooth thickness.

It is of interest to note that the same computer simulation can be used to establish realistic tolerances for master gears. The procedure is the same, except that, instead of tooth number, the diametral pitch of the master gear is altered. The resulting change in diametral pitch is then converted into a change in base pitch, which in turn is converted into tolerances for circular pitch and profile.

And it is worth noting that the same computer simulation can be used to determine the tight-mesh center distance for a gear pair with different thermal expansions. At elevated temperatures, for instance, the diametral pitch of a metal gear will be different from that of a mating plastic gear, which means that the equations for tight-mesh center distance are not valid.

Alternative to Master Gears

From the standpoint of the designer, the master gear method leaves much to be desired. Aside from the need for a variety of tooth numbers and face widths (usually not available), there is the aforementioned unknown error in measured tooth thickness. Also, there is the uncertainty in kickout between mating gears — as is evident from the following illustration. Consider a 20*P* master gear, two 19.95*P* gears, and two 20.05*P* gears, all perfect and with the same tooth number. The kickout between the 20*P* and the 19.95*P* will be about the same as between the 20*P* and the 20.05*P*. But there will be no kickout between the 19.95*P* gears, nor between the 20.05*P* gears. And the kickout between the 19.95*P* and 20.05*P* will approach the sum of their kickouts against the 20*P*. In short, the designer has no way of knowing how the kickouts will combine.

The unknown error in measured tooth thickness and the uncertainty in kickout can be avoided when the tooth thickness of one or both members of a gear pair is readily adjustable. For instance, when one member is a hobbed pinion and the other a formed gear (molded plastic, die cast, powder metal, stamped), the hobbing machine operator can simulate the assembly process by drawing parts at random from the gear lot and using them to check the pinions. The pinion specification would read: ROLL TEST CENTER DISTANCE WITH ANY MATING GEAR X.XXXX/X.XXXX, and KICKOUT WITH ANY MATING GEAR .XXXX. Also, because all teeth are somewhat unsymmetrical and misaligned, the pinion drawing would identify the mating flanks and active face width.

Roll Test Center Distance

Equations for roll test center distance are derived from Eqs. 3 and 4. That is, for $B = 0$ in Eq. 4, the

$$\text{inv}\ \phi = \text{inv}\ \Phi + \Delta T + \Delta t\ ,\ \frac{\Delta T + \Delta t\ ,}{2\ C_b} \qquad (7)$$

and from Eq. 3,

$$C = C_b \frac{\cos \Phi}{\cos \phi}. \qquad (8)$$

Because the total composite variation (also known as total composite tolerance, total composite error, and to a lesser extent as *rollout*) must fall within the limits of roll test center distance, and since Eqs. 7 and 8 are valid only for perfect teeth, it follows that the corresponding limits of ΔT and Δt are the limits of perfect teeth. In other words, the rollout must fall within the size tolerance.

It is pertinent to note that the German equivalents of rollout and kickout are also single words, namely, *Wälzfehler* (roll error) and *Wälzsprung* (roll jump), respectively.

From Eqs. 7 and 8 it is seen that inv ϕ must be converted into cos ϕ. This conversion can be easily carried out on programmable pocket calculators that sell for as little as $30 (Ref.11). Even so, the test radius method, which was devised to bypass Eqs. 7 and 8, is still in widespread use.

Test Radius Method

Originally, test radii were obtained by simply adding $\Delta t/(2 \tan \Phi)$ and $\Delta T/(2 \tan \Phi)$ to the reference radii of the work gear and master gear, respectively. But, since $(\Delta t + \Delta T)/(2 \tan \Phi)$ is just the first term of an infinite series (Ref. 12) that method was subject to a significant error.

Nowadays, the test radius of the work gear is defined as the radial distance from the center of the work gear to the reference circle of the master gear, as obtained by setting $\Delta T = 0$ (in Eq. 7) and subtracting the reference radius of the master gear from the center distance (Eq. 8).

It is of course necessary to set $\Delta T = 0$ in Eq. 7, since the test radius of the master gear is sill obtained in the original way, i.e., by adding $\Delta T/(2 \tan \Phi)$ to the reference radius. As a result, the designer is forced to specify the master gear tooth number (or range of tooth numbers), since the test radius of the work gear varies with number of teeth on the master gear. (The range usually narrows to a single number for large values of Δt.) In addition, the designer is forced to specify a tolerance on master gear tooth thickness, since the error in work gear tooth thickness varies with master gear tooth thickness.

For example, given an 8*T*-20*P*-20° pinion whose tooth thickness is 0.02830 inches greater than the basic tooth thickness (to eliminate undercut), and a 40*T* master gear.

Accordingly, in Eq. 7 the $\Delta t = 0.02830$, $\Delta T = 0$, $n = 8$, $N = 40$, $P = 20$, $C_b = (n + N)/2P = 1.2$, and $\Phi = 20°$. Then, from Eq. 8 the $C = 1.2353$, so the pinion test radius is $1.2353 - N/2P = 0.2353$. (For a 30T master gear, the pinion test radius would be 0.2346, not 0.2353.) Next, given that the master gear tooth thickness is 0.00080 inches greater than the basic tooth thickness ($\Delta T = 0.00080$), the master gear test radius is $N/2P + \Delta T/(2 \tan \Phi) = 1.0011$. Consequently, the roll tester is set to $0.2353 + 1.0011 = 1.2364$ inches.

But for this center distance the pinion tooth enlargement will not be $\Delta t = 0.02830$. Instead, for $C = 1.2364$ in Eq. 3 and $\Delta T = 0.00080$ in Eq. 6, the $\Delta t = 0.02845$, which is in error by 0.00015 inches. Because this error is a biased error (not a random error), "good gaging practice" dictates that it not be greater than about 5% of the tooth thickness tolerance. In other words, in this particular case the test radius method would not be valid if the tooth thickness tolerance were less than about 0.0030 inches.

Because the constraints on master gear tooth thickness and tooth number can be rather severe, the test radius method tends to raise the cost of manufacture. Another specification that tends to raise the cost of manufacture is rollout — a holdover from the days when rollout did not fall within the limits of test radius (Ref. 13). When not specified, rollout is free to increase as size variation decreases, and whether the backlash is caused by rollout or size variation is usually of no consequence.

A rollout specification is needed only in special cases, such as in high-speed gearing and in various type of anti-backlash gearing. A rollout specification is not an effective check for index error, since index error can be not only large for a small rollout (Refs. 14–15), but also small for a large rollout (as when the rollout opposes the accumulated spacing error).

Backlash Calculations

Assigning numbers to the center distance (C in Eq. 3) and to the sum of tooth thicknesses ($\Delta T + \Delta t$ in Eq. 4) is not an easy matter. The task would be fairly simple if the axes of rotation were parallel, but housing bores are not coaxial, bearing clearances are not equal, journals are not coaxial, runouts of bearing races are neither equal nor in phase, and mountings for gears on motors, encoders, etc. are tilted, as are the fixed studs on which gears are mounted. Furthermore, tooth reactions cause unequal radial shifts within the bearing clearances.

The center distance and the increase in the sum of tooth thicknesses are obtained by projecting both axes of rotation onto the axial and pitch planes, respectively. (The axial plane contains the nominal positions of both housing axes. The pitch plane is perpendicular to the axial plane and parallel to the housing axes.) In the axial plane the distance between the skewed axes at the point where the gears make contact is the center distance. In the pitch plane the product of the angle between the axes (in radians) and the smaller of the two face widths is the effective increase in the sum of tooth thicknesses.

If manufacturing distributions for all pertinent dimensions are available, then the backlash distribution can be obtained by simulating the assembly process on a computer (Refs. 16–17). Parts are drawn at random, assembled on the computer, and then the backlash (B in Eq. 4) is computed for each and every assembly. This method does not involve mathematical statistics. Moreover, it can handle any type of frequency distribution (bimodal, rectangular, skewed, decentered, sorted, small quantities).

If manufacturing distributions are not available (as when it is not cost-effective to ascertain and control both the shape and centrality of each distribution), then only the maximum backlash and minimum backlash can be obtained. Specifically, maximum backlash is that for maximum center distance and minimum tooth thicknesses. Minimum backlash is that for minimum center distance and maximum tooth thicknesses, plus the effective increase in the sum of tooth thicknesses due to misalignment.

The minimum backlash can be slightly negative (a potential, but improbable interference), since the range of the backlash distribution will be less than the difference between maximum backlash and minimum backlash.

The magnitude of the potential interference is usually dependent upon the manufacturing process. In particular, parts made with hard tooling (molded plastic, die cast, powder metal, stamped, cold-drawn) tend to have decentered distributions, so the potential interference cannot be as great as for parts whose dimensions are readily adjustable during the course of manufacture.

In most designs the potential interference can range from 15 to 25 percent of the difference between the maximum backlash and minimum backlash, depending upon the shape and centrality of the various distributions.

Errors in Roll Testing

The two-flank roll test has always been plagued by significant errors in measurement. For example, about 30 years ago Michalec and Karsch conducted a correlation study (Ref. 18), wherein a number of companies inspected a large assortment of precision gears for test radius, rollout and kickout. The discrepancies between the participants were indeed significant. For instance, the average of all the greatest discrepancies for test radius (each gear had a greatest discrepancy, namely, the difference between the highest and lowest reported measurements) was an incredible 0.0016 inches!

If a similar correlation study were to be conducted today, the discrepancies probably would be similar, since there have been no marked improvements in inspection practice and test equipment. Techniques for preventing significant errors in measurement have been known for some time (Refs. 19–20), but they are seldom implemented.

To insure against significant errors in measurement, the designer can, in self-defense, invoke a process specification (as is done for heat treating, application of dry-film lubricants, etc.) that spells out the roll test requisites, namely, mounting method, axis alignment, permissible hysteresis and nonlinearity of the movement, master gear dimensions and tolerances, and how to determine the checking load, checking speed, load correction and temperature correction. Without an effective process specification, the designer can do little more than cross his/her fingers and hope for the best. ■

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