BACK TO BASICS . . .

Functions of Gearing and Application of the Involute to Gear Teeth

by
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When it is required to transmit rotary motion from one shaft to another at a definite ratio, with the shafts rotating in opposite directions, several means can be employed. One method is to use friction disks, as shown at *A*, Fig. 1.

Pulleys and a crossed belt can also be used, as shown at *B*, Fig. 1. If, however, uniform angular motion must be transmitted, neither the friction disks, nor the pulleys and crossed belt, will prove satisfactory.

Friction disks cannot be relied upon to transmit uniform angular motion, because they lack positive driving contact. As soon as sufficient load is imposed on the driven member, slippage between the disks occurs.

Pulleys and a crossed belt are more dependable than the friction disks due to the greater area of frictional contact, but here again, because of the lack of positive driving contact between the crossed belt and the pulleys, belt slippage occurs when sufficient load is applied to the driven pulley. Therefore, uniform angular motion cannot be maintained by pulleys and a crossed belt.

Gears, as shown at *C*, Fig. 1, offer the most practical and dependable means for transmitting uniform angular motion, but, the shape of the teeth has an important bearing on the smoothness of the motion transmitted.

Experience has proven that the involute provides the most satisfactory profile for spur and helical gear teeth, and fulfills the requirements for transmitting smooth uniform angular motion.

The Involute

The involute can be described as that curve traced by a point on a crossed belt as it moves from one pulley to the other without slippage. This can be demonstrated by attaching a template to the driven pulley, as shown at *A*, Fig. 2. Rotation of the driver pulley will cause the tracing point on the crossed belt to trace an involute on the template attached to the driven pulley.

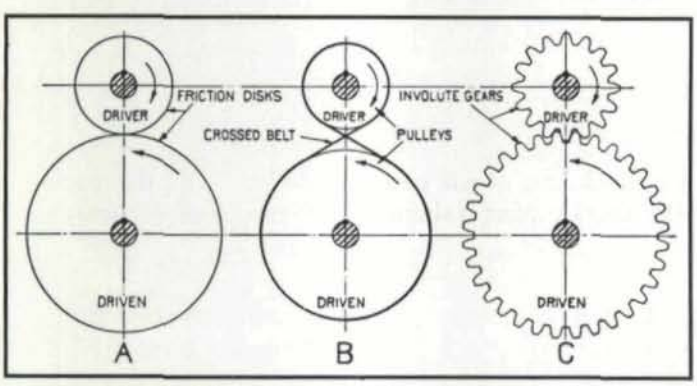

Fig. 1 — Diagram Illustrating Three Means for Transmitting Rotary Motion from One Shaft to Another.

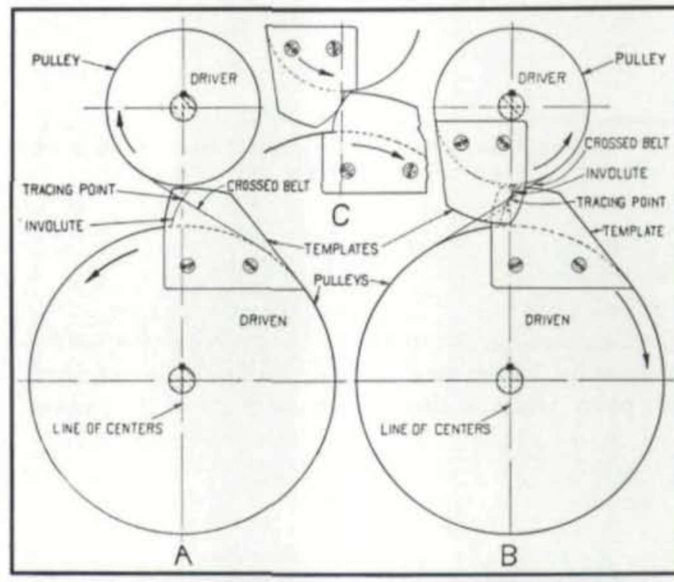

Fig. 2 — Diagram Illustrating That the Involute Is That Curve Traced by a Point on a Crossed Belt as It Moves from One Pulley to Another.

Similarly, a mating involute can be developed by attaching a template to the driver pulley, as shown at *B*, Fig. 2. A tracing point on the other side of the crossed belt will trace an involute on this template when the driver pulley is rotated in the opposite direction.

If these traced templates are cut to the traced curves, as shown at *C*, in Fig. 2, they produce mating involute profiles, which will transmit uniform angular motion, when the connecting belt is removed.

Since each of these profiles is only one side of a single involute gear tooth, they will only transmit motion through a slight arc of rotation. To transmit continuous uniform angular motion, succeeding equally-spaced and parallel profiles must be provided. This has been done, as shown at *A* in Fig. 3, where equally-spaced tracing points *a* have been located on the crossed belt.

If motion is to be transmitted in both directions, then similar profiles must be provided, as shown at *B* in Fig. 3, where tracing points *b* on the other side of the crossed belt have been added. We now have developed involute gear teeth to transmit uniform angular motion which replace the friction disks, pulleys and crossed belt.

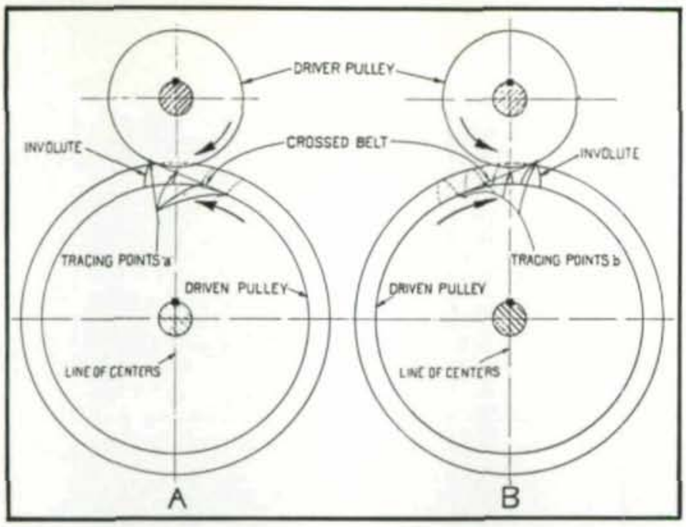

Fig. 3 — Diagram Showing How Equally-spaced Points on Crossed Belt Trace Successive Equally-spaced and Parallel Involute Profiles.

In Fig. 4, the gears, pulleys and crossed belt have been super-imposed on the friction disks. Here, it will be noticed that the friction disks are the pitch circles of the gears. The pulleys are the base circles, and the crossed belt represents the line of action. Where the belt crosses the line of centers is the pitch point, which is also the tangent point of the friction disks, or the common tangent. The angle between the crossed belt, or line of action, and the common tangent is called the pressure angle.

Changes in Center Distance Do Not Affect Velocity Ratio of Conjugate Tooth Action

A cardinal virtue of the involute is that it permits a change in center distance without affecting the velocity ratio, or conjugate tooth action. When the center distance is increased, as shown at *B* in Fig. 5, the base circles remain the same but a new line of action, new pitch circles and a new pitch point are created. All of these elements, when located at the standard center distant at *A*, are shown in full lines; and the same elements in dotted lines, when the center distance is increased, as shown at *B*. It will also be noted that the operating pressure angle is increased when the center distance is increased; and, as will be subsequently shown, the length of the line of contact is shortened if the tooth lengths remain the same.

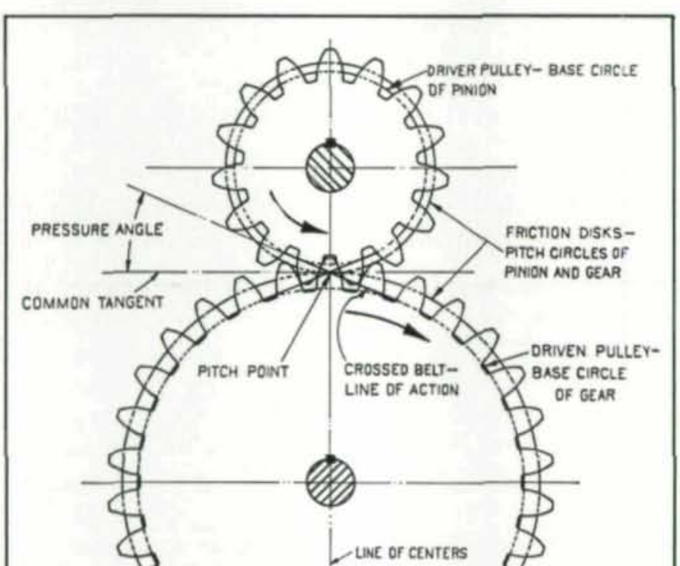

Fig. 4 — Diagram Showing Gears, Pulleys and Crossed Belt, Superimposed on Friction Disks, Illustrating Basic Gear Elements.

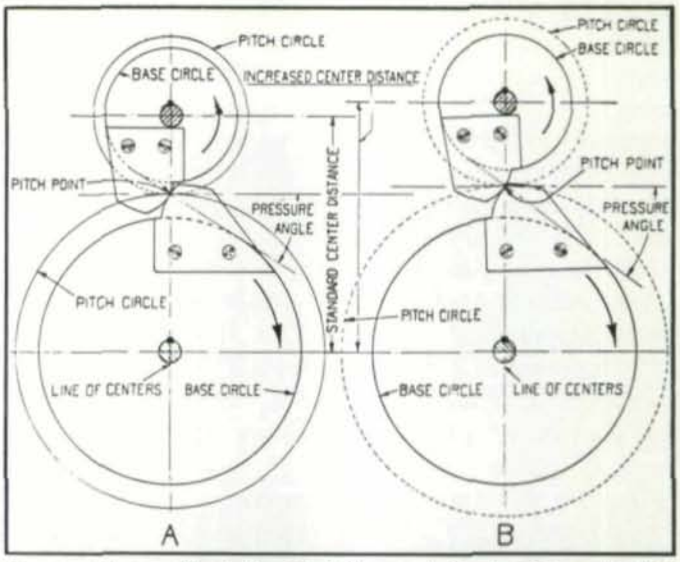

Fig. 5 — Diagram Illustrating That Changes in Center Distance Do Not Affect Velocity Ratio, or Conjugate Tooth Action.

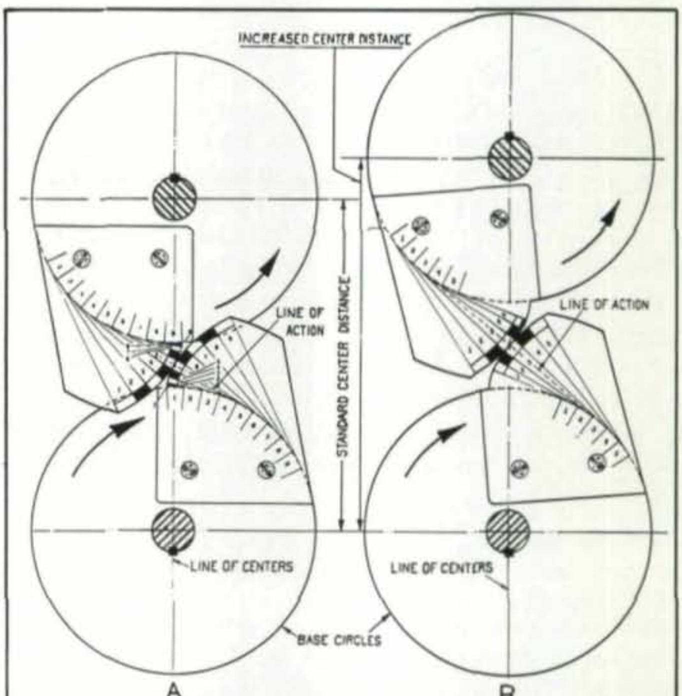

Fig. 6 — Diagram Illustrating Relative Slippage of Involute Gear Teeth.

Gear Tooth Slippage

A study of the action of mating involute profiles shows that they slide upon each other for their entire contact. At *A* in Fig. 6, both base circles are divided into an equal number of spaces, and tangent lines are drawn from the division points on the base circles to the involute profiles.

It will be noted that while the divisions of the base circles are equal, the corresponding arcs on the involute profiles are unequal, increasing in length as they depart from the base circles. Corresponding divisions on the base circles and involute profiles are numbered for comparison and identification, and indicate those portions of the profiles that are in contact with each other during equal pitch line movements of the gears.

The relative lengths of the mating arcs on the profiles is a measure of the amount of slippage between the profiles. The greatest amount of slippage takes place when the end of one profile is in contact with the mating profile in the vicinity of the base circle. It will also be noted, that involute contact takes place on a line tangent to both base circles—*this is the line of action.*

When the center distance is increased, the relative rate of slippage is not so great as before, as shown at *B* in Fig. 6. The reason for this is that the contact between the involute profiles takes place farther away from the base circles.

The Involute Rack

The parent rack of the involute system of gearing has straight sides. In Fig. 7, the involute template representing one side of a gear tooth is shown contacting one side of a rack tooth and driving the latter with a uniform linear motion. Note that when the involute template has moved distance *Y* on the line of action, the rack tooth has moved distance *X* in a linear direction. If distance *Y* is the base pitch of the gear—the circular pitch transferred to the base circle—and distance *X* is the linear pitch of the rack, then equal angular rotations of the gear must transmit equal linear movements to the rack, because the circular pitch of the gear must equal the linear pitch of the mating rack. It should be noted that the face of the rack tooth must be at right angles to the line of action.

Basic Principles of Involute Gearing

With properly designed and generated involute gears, smooth continuous action results, but improper design can defeat the obtainment of this objective. The problem of designing gears can be greatly simplified if we thoroughly understand the basic principles involved. There are three basic elements in the design of involute gearing, *viz.*: center distance, base circle diameter and tooth ratio.

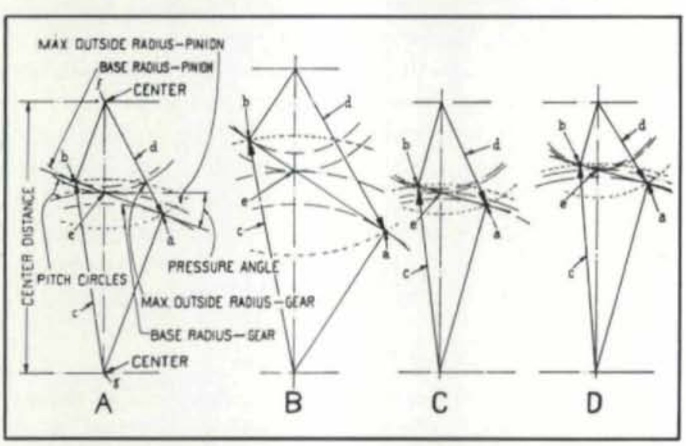

Fig. 7 — Diagram Illustrating That a Straight-sided Rack Tooth is the Proper Shape to Engage with an Involute Gear Tooth—Also That an Involute Contacting a Rack Transmits Uniform Linear Motion to the Rack.

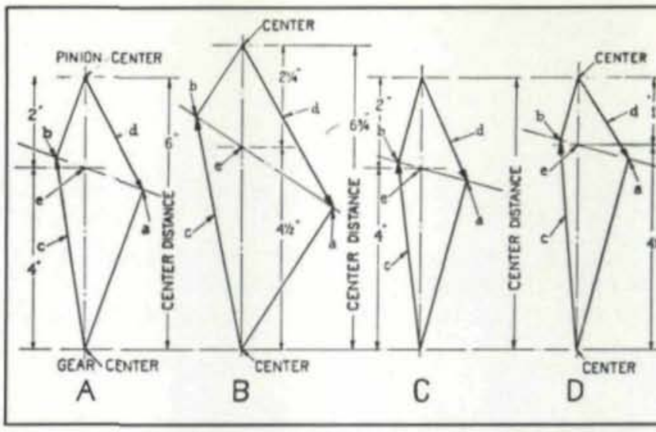

Fig. 8 — Diagram Illustrating Basic Elements of Involute Gearing.

These fundamental conditions or requirements can be clearly illustrated and explained by constructing simple diagrams, as shown at *A, B, C* and *D*, Fig. 8. To proceed: draw a center line, as shown at *A*. Space off on this line the required center distance. The ratio of the two gears is then spaced off, as represented by point *e*, which is the pitch point of the pair of gears. An angular line drawn through pitch point *e* represents the line of action.

Lines drawn from the centers of gear and pinion, and at right angles to the line of action, locate points *a* and *b* which represent the origins of the involutes, or the base radii of gear and pinion, respectively. Now draw lines *c* and *d* from both centers to points *a* and *b* on the line of action. Points *a* and *b*, in addition to representing the points of origin of the involutes, are also the interference points, as will be explained later. Radius *c* represents the maximum permissible outside radius of the gear, and radius *d*, the maximum permissible outside radius of the pinion.

Now notice what happens to the diagram at *A*, when the center distance is increased as shown at *B*, the base radii and ratio remaining the same as at *A*. First, the pitch point *e* is located at a different position. Locations of the interference points *a* and *b* on the line of action have changed, affecting the maximum permissible outside radii of gear and pinion, respectively. The pressure angle has also increased. This increase in the center distance, as has been previously ex-

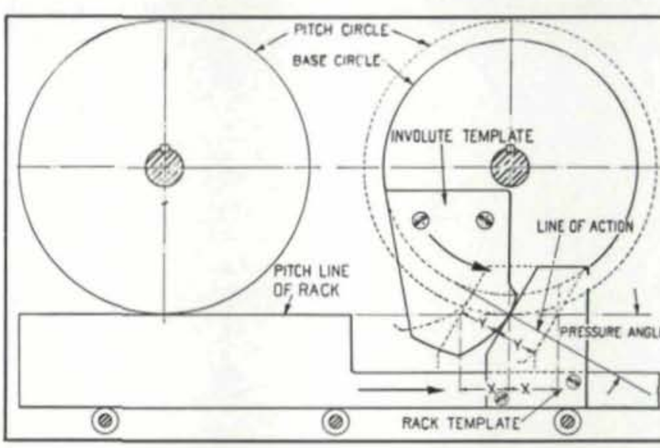

Fig. 9 — Similar Diagrams to Fig. 8, Except That Arcs Have Been Added to Indicate Pitch Circles, Base Circles, and Maximum Permissible Outside Radii of Gear and Pinion, Respectively.

plained, will not necessarily affect conjugate tooth action, but it does change the relative contact positions of the tooth profiles, as indicated by the tooth slippage diagrams in Fig. 6.

At *C*, in Fig. 8, the center distance and tooth ratio remain the same as at *A*, but the angle of inclination of the line of action has been reduced. Note that this affects the base radii of gear and pinion, and also the maximum permissible outside radii of gear and pinion.

At *D* in Fig. 8, the inclination of the line of action is the same as at *C*, but the ratio has been changed. This change affects the outside and base radii of gear and pinion, respectively.

At *A, B, C* and *D*, Fig. 9, diagrams similar to those in Fig. 8 have been constructed, with the addition of arcs passing through points, *a*, *b* and *e*. It will be noted that these arcs establish, in each case, the base circles, pitch circles and maximum outside circles of gear and pinion, respectively. A comparison of these diagrams clearly demonstrates the interrelationship of the three fundamental elements of gear design, *viz.*: center distance, base circles and tooth ratio.

It has been demonstrated that a change in the base circle center distance affects the operating pressure angle of two involute curves developed from these base circles. This is not the case when an involute gear tooth and rack are brought into contact. It has been demonstrated in connection with Fig. 7 that the line of action must lie at right angles to the rack tooth face. At *A* in Fig. 10 a gear and rack tooth are shown in mesh at distance *a*. At *B*, Fig. 10, this distance *a*, as indicated by distance *b*, is shown appreciably increased.

In order to keep the line of action at right angles to the rack face, the pitch point has moved a distance equal to *c*, and thus established a new position for the rack pitch line. It will also be noted that increasing the distance of the base circle center has not affected the base circle, or the pitch circle of the involute gear tooth, nor has it changed the operating pressure angle.

From the foregoing, we may make the following conclusions concerning the involute curve and its application to gears operating on parallel axes.

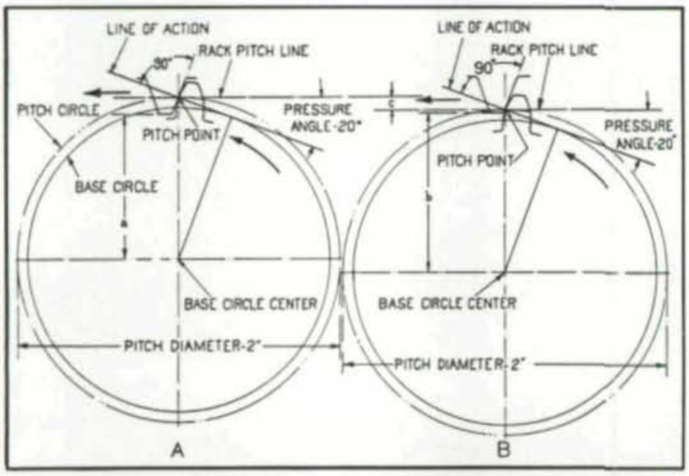

Fig. 10 — Diagram Illustrating That the Pressure Angle and Pitch Circle Diameter Do Not Change When a Gear Tooth Is Moved Closer to or Farther Away from a Rack.

Fundamental Laws of the Involute Curve

1. The involute is wholly determined by the diameter of its base circle.

2. An involute moving about its base circle center imparts rotative motion to a contacting involute in the exact ratio of the diameters of their respective base circles.

3. An involute has no pressure angle until brought into intimate contact with another involute or a rack.

4. The pressure angle is determined by the center distance and the base circle diameters.

5. The pressure angle once established is constant for a fixed center distance.

6. An involute has no pitch diameter until brought into intimate contact with another involute or a rack.

7. The pitch diameter of an involute contacting another involute is determined by the center distance and the ratio.

8. The pressure angle of an involute contacting a rack is unchanged when the base circle center is moved toward or away from the rack.

9. The pitch diameter of an involute contacting a rack is unchanged when the base circle center is moved toward or away from the rack.

10. The pitch line position of an engaging involute and rack is determined by the intersection of the line of action and a line passing through the base circle center and perpendicular to the direction of rack travel.

Gear Tooth Interference

The term gear tooth interference is defined as contact between mating teeth at some other point than along the line of action. Two types of interference are sometimes encountered. One is known as involute interference, and the other as fillet interference. Involute interference is the term used to indicate that contact between mating teeth takes palce at some other point than on the line of action *outside* the zone of contact. Fillet interference refers to contact at some other point than on the line of action *inside* the zone of contact. The zone of contact is that portion of the line of action bounded by the "natural" interference points, see Fig. 11.

Involute Interference

The condition known as involute interference is sometimes encountered when the mating gear teeth lock, or will not pass freely through mesh. Fig. 11 illustrates diagrammatically a pair of gears the outside circles of which are enlarged to the interference points. It will be noted that the tooth length on the smaller gear can be increased appreciably more than that on the larger gear without interference. Hence, it is the teeth on the larger gear which cause the trouble. If the ends of the teeth on the larger gear extend beyond the "natural" interference point, non-involute contact results, because the teeth on the gear contact the non-involute flank of the pinion teeth and thus prevent free rotation. At *A* in Fig. 12, it will be noted that the outside circle of the teeth on the larger gear extends considerably beyond the interference point, and hence interferes with the flanks of the teeth on the smaller gear. This condition has been exaggerated to illustrate the term known as "involute interference." If the flanks on the pinion teeth are radial, or extend inside of a radial line tan-

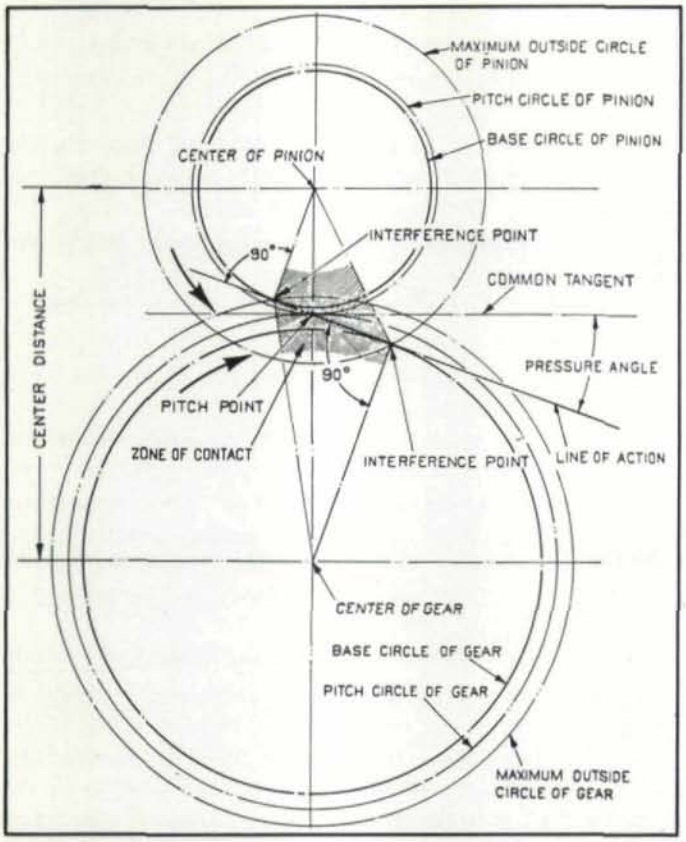

Fig. 11 — Diagram Presenting Principal Boundary Surfaces of Gear and Pinion, Showing Maximum Permissible Outside Diameters to Avoid Involute Interference.

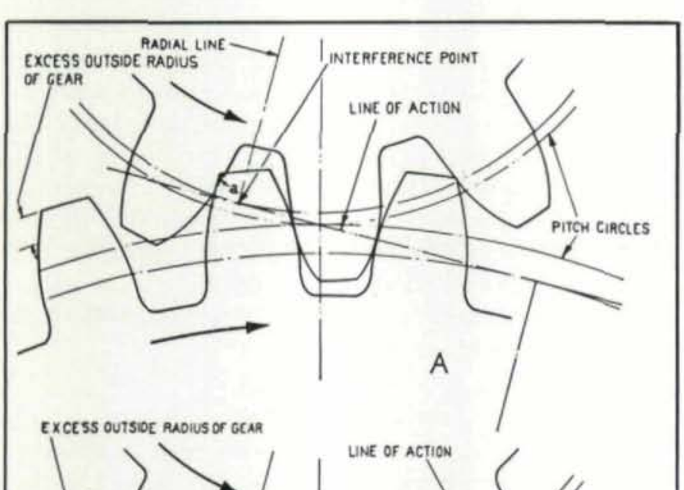

Fig. 12 — Diagram Illustrating Term Known as Involute Interference. Gear Tooth Contacts Non-Involute Portion of Pinion Tooth Outside the Zone of Contact.

gent to the involute at the base circle, the amount of interference would be greater than if the flanks were undercut. The amount of undercut is governed by the type of

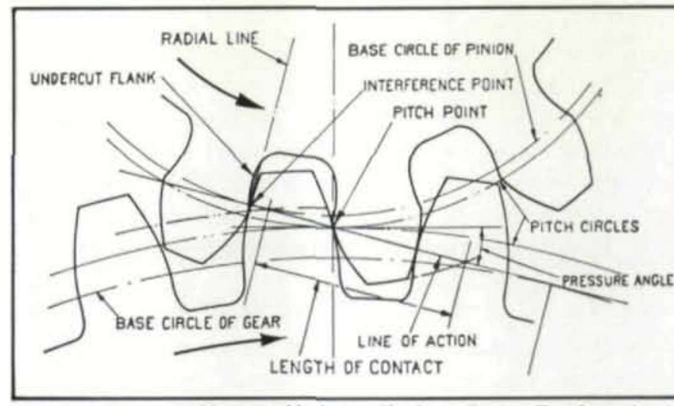

Fig. 13 — Diagram Showing Undercut Flank on Pinion Tooth to Avoid Involute Interference.

cutter used, the ratio between the number of teeth on the pinion and cutter, and the pressure angle. The cross-sectioned portion *a* of the tooth at *A*, Fig. 12, shows that the gear tooth overlaps the pinion tooth, and thus prevents free rotation.

At *B*, Fig. 12, the gears are rotated into a different position to illustrate two points of interference at *b* and *c*. It is evident that these gears cannot be rotated in either direction, or in other words they "lock," and free rotation is impossible.

Involute interference can be avoided in several different ways. One method is to design the cutter so that it removes the interfering portions of the pinion teeth, as shown in Fig. 13. This method is objectionable, because, as will be noted, it undercuts the flanks of the pinion teeth, reducing their strength. Furthermore, if the undercutting is excessive and extends above the base circle, it shortens the active involute portion of the teeth, reducing the length of the line of contact, and may result in lack of continuous action.

Another method is to use enlarged and reduced diameters on pinion and gear, respectively, or what is known as long- and short-addendum teeth. This particular method is recommended when the pinion has a comparatively small number of teeth and meshes with a gear having three or more times as many teeth; also when a small pressure angle is used, and standard center distance must be maintained. This method is illustrated diagrammatically in Fig. 14, where the outside diameter of the pinion is enlarged, and the outside diameter of the gear reduced the same amount. In Fig. 14, it will be noted that the original outside circle of the gear, indicated by

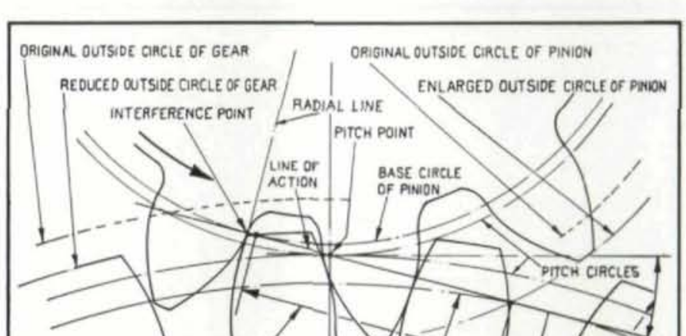

Fig. 14 — Diagram Illustrating Application of Long- and Short-Addendum Teeth to Avoid Involute Interference.

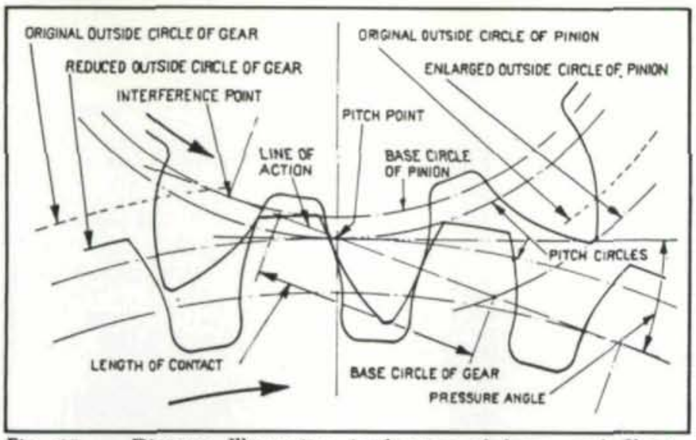

Fig. 15 — Diagram Illustrating Application of Long- and Short-Addendum, Coupled with Increased Pressure Angle to Avoid Involute Interference.

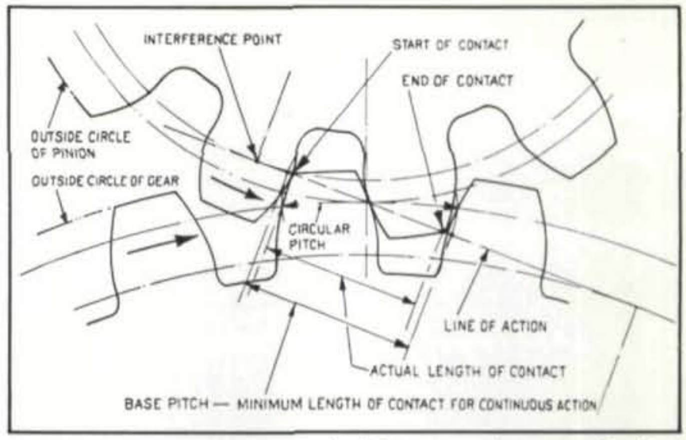

Fig. 17 — Diagram Illustrating Lack of Continuous Action as a Result of Shortened Teeth.

the dotted lines, extends beyond the interference point on the pinion teeth. By shortening the addendum on the gear tooth interference is avoided, and by lengthening the pinion addendum, length of contact is increased.

A comparison of the diagrams in Figs. 13 and 14 representing a pair of gears of the same tooth ratio, and designed to operate at standard center distance, shows that with the undercut condition in the flank of the pinion tooth, the length of contact is shorter than in Fig. 14, where long- and short-addendum teeth are used. In fact, the length of the line of contact is increased 30% in the case of the long-and short-addendum teeth.

Still another suggestion is to maintain the same reduction and enlargement of gear and pinion diameters, and increase the pressure angle. This condition is presented diagrammatically in Fig. 15. Here it will be noticed that the start and end of contact lie well inside the interference points, and that the length of contact is about 15% less than in Fig. 14. Therefore, an increase in the pressure angle coupled with long- and short-addendum teeth keeps contact on both gear and pinion teeth farther away from the interference points. When severe cases of interference are encountered, this method is recommended.

Fillet Interference

The condition in gear operation known as fillet interference is illustrated diagrammatically in Fig. 16. This occurs when

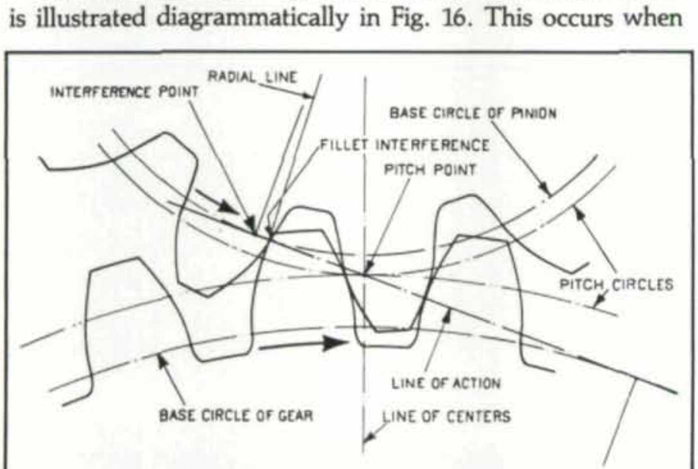

Fig. 16 — Diagram Illustrating Term Known as Fillet Interference. Gear Tooth Contacts Non-Involute Portion of Pinion Tooth Inside the Zone of Contact.

the flank of the pinion tooth below the base circle lies inside the radial line tangent to the involute at the base circle. This condition prevents free rotation of the teeth and the transmission of uniform angular velocity. There are several solutions that can be adopted to avoid fillet interference. One is to enlarge the pinion diameter and reduce the mating gear diameter a similar amount, or in other words use long and short addendum teeth as explained more fully in Chapter II. Another method is to design the cutter so that it will remove the interfering portion in the flank of the tooth. This method might result in shortening the length of the line of contact, especially if the undercut extends beyond the base circle towards the pitch circle.

Lack of Continuous Action Due to Shortened Teeth

This term refers to a condition in gear design, where the actual length of tooth contact is less than the base pitch. In other words, one pair of teeth go out of contact before a succeeding pair are in position to make contact, resulting in

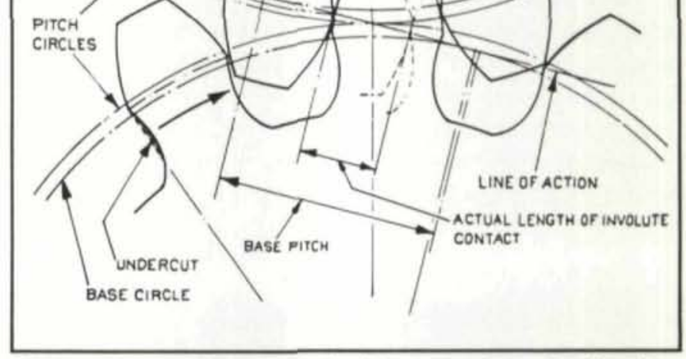

Fig. 18 — Diagram Illustrating Lack of Continuous Action Resulting from Undercutting of Flanks of Teeth. Also Due to Small Pressure Angle and Small Number of Teeth in Both Members.

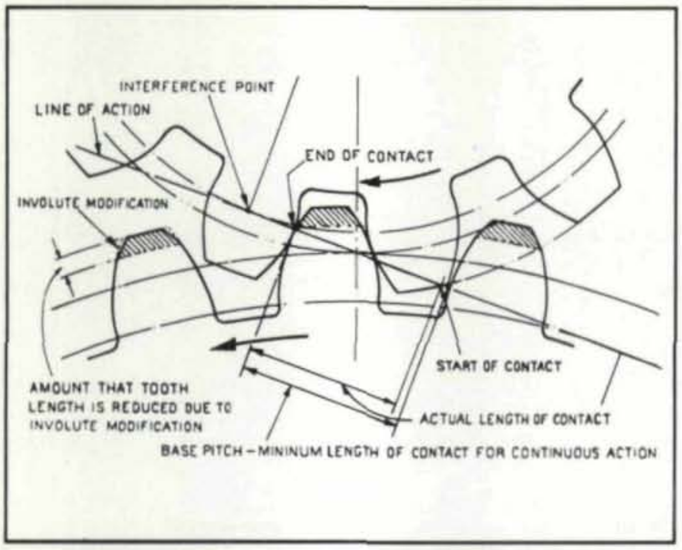

Fig. 19 — Diagram Illustrating Lack of Continuous Action Resulting from Involute Modification.

interrupted action. Fig. 17 shows diagrammatically a pair of gears in which the teeth are shortened to such an extent that they lack continuous action, and hence fail to transmit uniform angular motion. It will be noticed that the base pitch, which is the circular pitch transferred to the base circle is greater in length than the actual length of contact.

Lack of Continuous Action Due to Undercut

Fig. 18 shows another condition that can result in lack of continuous action. Here two 12-tooth pinions are shown in mesh, and it will be seen that the actual length of involute contact falls far short of meeting the requirements for continuous action. There are two reasons for this. One is the small number of teeth in the mating gears, and the other is the small pressure angle. The small pressure angle results in a severe undercutting of the flanks of the teeth, and the small number of teeth, of course, reduces the number of teeth in contact.

Lack of Continuous Action Due to Tooth Modification

Another condition that can cause lack of continuous action is involute modification. Fig. 19 illustrates a pair of gears, the tooth shape on one of these gears being modified near the tip. In this case, true involute contact does not start at the tip of the tooth on one gear, and hence, falls short of complete profile contact. Under light loading conditions, these gears might fail for continuous action, because of the shortened tooth profile above the pitch circle; but under heavy loading, the contact could extend over a greater length of the tooth profile. Gears are sometimes cut "off" pressure angle to compensate for tooth deflection under heavily loaded conditions; also to take care of tooth distortion due to heat treatment.

DETERMINATION OF GEAR RATIOS . . .

(Continued from page 36)

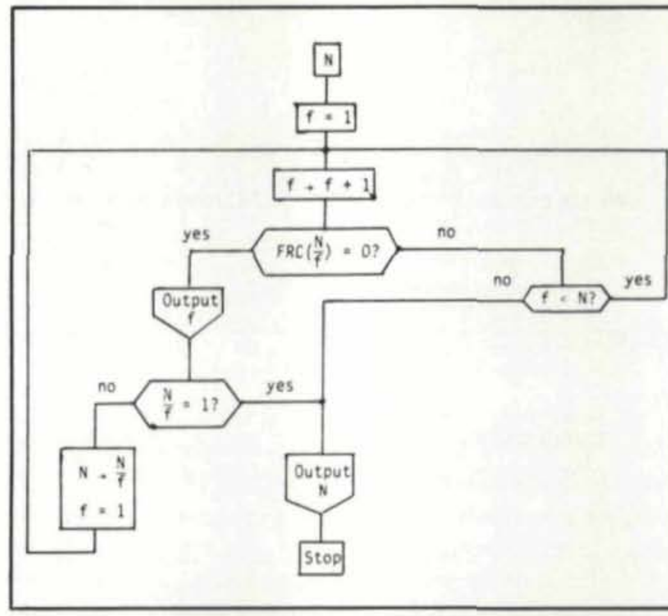

FIG. 2—Flowchart for FACTOR program

Example 3. Design a gear train to provide an angular velocity ratio of $\sqrt{5}$. Computer output is displayed in Table 1 along with the computation times using program GEAR RATIO followed by successive uses of REVISE.

Use of a factoring routine once again for $N_1 = 2889$ and $N_2 = 1292$ yields a four gear set with

$$R = \frac{27}{19} \times \frac{107}{68} = 2.2360681.$$

Thus, $N_1 = 19$, $N_2 = 27$, $N_3 = 68$, and $N_4 = 107$ teeth. Obtaining still greater accuracy is more difficult from a practical viewpoint because 13 and 421 are the only factors of 5473 while 47 and 1103 are the only factors of 51,841.

The flowchart for a typical factoring routine is shown in Fig. 2.

Table 1 Rational numbers approximating $\sqrt{5}$

E	p_N	q_N	Approx. computation* time (s)
1×10^{-2}	38	17	4
1×10^{-3}	38	17	2
1×10^{-4}	161	72	3
1×10^{-5}	682	305	4
1×10^{-6}	2889	1292	3
1×10^{-7}	12,238	5473	4
1×10^{-8}	12,238	5473	5
1×10^{-9}	115,920	51,841	5

* The first calculation used the GEAR RATIO program and the remaining calculations were performed by entering the program at REVISE. Direct calculation of the last ratio (115 920/51841) using GEAR RATIO required approximately 18 s, rather than the 30 required for the step-by-step calculations.

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