Accurate and Fast Gear Trigonometry

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Summary:

An accurate and fast calculation method is developed to determine the value of a trigonometric function if the value of another trigonometric function is given. Some examples of conversion procedures for well-known functions in gear geometry are presented, with data for accuracy and computing time. For the development of such procedures the complete text of a computer program is included.

Existing Complications

The many trigonometric functions in gear geometry may cause peculiar complications in the calculation of quantities. Very often the value of a certain function is known, and the value of another function has to be calculated, sometimes with available inverse functions, and sometimes without any other help other than laborious procedures. Such a complication already occurs in the first application of the definition of an involute.

An involute is defined on a base circle with fixed radius r_b. The polar coordinates of a point of an involute are

$$\frac{r_b}{\cos\alpha_t} \text{ and } \mathrm{inv}\alpha_t$$

(See Fig. 1.) The pressure angle α_t is a parameter in the two functions $r_b/\cos\alpha_t$ and $\mathrm{inv}\alpha_t$. Leaving the fixed radius r_b out of consideration, the two functions to be examined are the secant, $1/\cos\alpha_t$, and the involute function $\mathrm{inv}\alpha_t$. A point on the involute is determined by the variable transverse pressure angle α_t or by one of the above functions. If one of these functions is known, the other can be calculated.

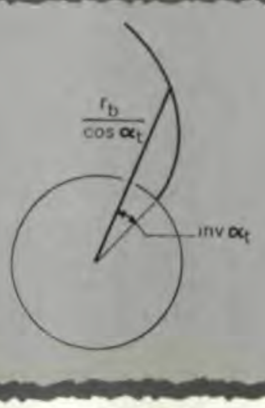

Fig. 1

From a given secant, $1/\cos\alpha_t$, the involute function can be determined easily in two steps.

$$\mathrm{TALT} = \tan\alpha_t = \sqrt{\left(\frac{1}{\cos\alpha_t}\right)^2 - 1} \quad (1)$$

$$\mathrm{inv}\alpha_t = \tan\alpha_t - \alpha_t = \mathrm{TALT} - \arctan(\mathrm{TALT}) \quad (2)$$

For instance, this sequence of calculations will be applied in the determination of the sum of addendum modification coefficients in a gear pair with given center distance. The square root and the inverse function, arctan, are standard functions in all computer languages.

The reverse calculation, starting with the involute function $\mathrm{inv}\alpha_t$ and finding the secant $1/\cos\alpha_t$, is of equal importance in gear calculations, but direct transcendental functions are not available in any computer language. Special iteration processes had to be written in each program. However, direct conversion procedures may be developed which are accurate, fast, and easy.

Common Calculation Methods

A common calculation method is an iteration process with three characteristics:

- *The introduction of one or two initial values,
- *The repeated use of an approximation formula or a set of formulae,
- *The procedural decision in each step based on the comparison of an intermediate result with an agreed tolerance value.

The *"regula falsi"* iteration method has two initial values, one below and the other above the expected solution. The approximation formula is a chord between two points, yielding a next point to be examined. The comparison decides whether the procedure can be stopped or with which points the procedure has to continue.

The "tangent" iteration method can be applied if a formula for the tangent of the function is known. Then only one initial value suffices. The approximation formula includes the tangent, yielding just one point to continue the procedure. The comparison only concerns the tolerance. An excellent example of such an iteration process[1-2] is

$$\phi_1 = (3.\ \text{inv}\alpha_t)^{0.330} \quad (3)$$

$$\phi_{j+1} = \phi_j + \frac{\text{inv}\alpha_t + \phi_j - \tan\phi_j}{\tan^2\phi_j} \quad (4)$$

$$j = 1,2,3,\ldots \text{ until abs}(\phi_{j+1} - \phi_j) < \text{tolerance} \quad (5)$$

$$1/\cos\alpha_t \approx 1/\cos\phi_{j+1} \quad (6)$$

In this process the involute function $\text{inv}\alpha_t$ is given, and the transverse pressure angle α_t is approximated, after which the secant can be calculated. The iteration formula (Equation 4) replaces the true function by its tangent. (See Fig. 2.) The initial value (Equation 3) may be as decisive for the

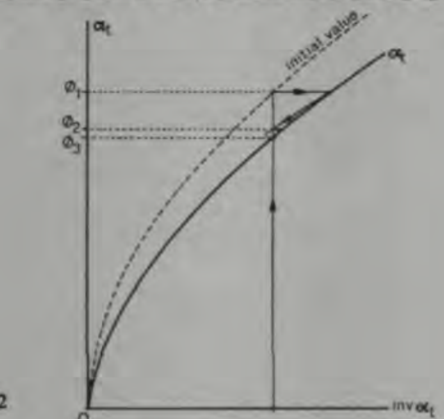

Fig. 2

number of steps as the agreed value of the tolerance. In this case the initial value was adequate and only by a trial-and-error numerical assay could it be slightly improved to

$$\phi_1 = (2.80.\text{inv}\alpha_t)^{0.333} \quad (7)$$

A significant improvement of the determination of the secant by any amendment in this iteration process could hardly be expected. Nevertheless, a development with a long history resulted in a new efficient procedure. In retrospect, the new method has a clear relation with the above iteration process, although "iteration" is abandoned.

A Promising Method

A very interesting method was presented by Jennings.[3] The main idea was to

*derive an expression to approximate the function for small values of the variables, and

*modify the expression to approximate larger values.

An expression suitable for small values is easily derived from the first terms in infinite series.

$$\tan\alpha_t = \alpha_t + \frac{1}{3}\alpha_t^3 + \ldots \quad (8)$$

$$\sec\alpha_t = \frac{1}{\cos\alpha_t} = 1 + \frac{1}{2}\alpha_t^2 + \ldots \quad (9)$$

For small values of α_t, Equations 8 and 9 simplify into

$$\text{inv}\alpha_t \approx \frac{1}{3}\alpha_t^3 \quad (10)$$

$$\sec\alpha_t - 1 \approx \frac{1}{2}\alpha_t^2 \quad (11)$$

Elimination of the right-hand terms yields

$$\sec\alpha_t \approx 1 + 1.0400(\text{inv}\alpha_t)^{2/3} \quad (12)$$

The approximation (Equation 12) does not hold for larger values, but Jennings was successful in manipulating a term $(\sec\alpha_t - 1)$ into (12) in such a way that it became a cubic equation with the root

$$\sec\alpha_t \approx 1 + 1.0400(\text{inv}\alpha_t)^{2/3}\{1 + 0.3082(\text{inv}\alpha_t)^{2/3}\} \quad (13)$$

The approximation (Equation 13) was introduced in gear calculations by Tuplin,[4] who recommended simple but effective methods preferably to be carried out with no other help than a slide rule. With respect to its simplicity, this approximation is remarkably accurate in a rather large range:

*For α_t between 0° and 53°, the approximation remains below the exact value, with a maximum error of −0.000252 for $\alpha_t = 46.2°$.

*Above $\alpha_t = 53°$, the approximation rises above the exact value with a rapidly increasing error.

Polynomial Method

The next step, References 5-8, applied in Reference 9, is obvious. Equation 13 looks like a series expansion in which the coefficients and the number of terms may be adapted to better accuracy. Such a development was not only needed for the move from the slide rule to the computer in a special case, but it also offers a general method for the conversion of several transcendental functions. The method follows.

*Determine a simple expression to approximate the function for small values of the variable. It may be an exponential expression inspired by the first terms of infinite series.

*Determine the range of the variable in which the result has to be sufficiently accurate.

*Apply a least square root method,[10-11] to establish a polynomial of a certain degree (concerns the number of terms in the formula) and with a certain number of decimal places (in the coefficients of the formula).

*Check the result and possibly select another degree or another number of decimal places.

The result of the above easy effort is an accurate and fast conversion procedure that can be applied in any computer program. Theoretically, it may be considered to be an improvement of the tangent iteration method, since it replaces the rectilinear tangent (one-degree polynomial) by a smooth curve (multi-degree polynomial). That smooth polynomial

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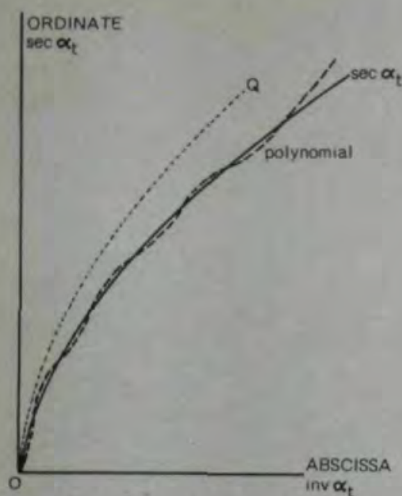

Fig. 3

CORRECTION

The IMTS-90 booth number for CIMA KANZAKI appeared incorrectly in the July/ August issue of GEAR TECHNOLOGY.

The correct booth number is 6192

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curve approximates the function so closely that "iteration" is unnecessary. (See Fig. 3.)

The conversion of an involute into a secant may serve as an example of this method. The infinite series (Equations 8 and 9) were already simplified into Equations 10 and 11. The coefficients in (10) and (11) can be neglected, since the polynomial will produce new coefficients. It suffices to determine the exponent of an auxiliary expression. In this example it is

$$Q = (\text{inv}\alpha_t)^{2/3} \tag{14}$$

The relevant input of the polynomial program is

*the involute function, referred to as ABSCISSA,
*the secant function, referred to as ORDINATE,

Table 1 – Polynomial Program

```
program POLYNOMIAL(input,output);
var
   AL,EXPNT,fl,h,p,pp,pl,Q,XABSC,xl,Y,YAPPR:
   real; a,alfa,b,beta,f,pn,pnl,x: array[0..17] of real;
   DCML,i,j,k,n,NMAX: integer;
function ABSCISSA: real;
begin ABSCISSA:=sin (AL)/cos(AL)-AL
end;
function ORDINATE: real;
begin ORDINATE:=1/cos(AL)
end;
function EXPONENT: real;
begin EXPONENT:=2/3
end;
function DECIMROUND(X:real): real;
var MM,XF,XR,XX: real; M,DR: integer;
begin XR:=abs(X); XF:=round(XR);
   XR:=XR-XF; DR:=DCML; MM: = 1;
   repeat M:=10*M; DR:=DR-1
   until (DR=0) or (M=1000);
   XX:=round(M*XR); XR:=M*XR-XX;
   MM:=MM*M; XF:=XF+XX/MM
   until DR<=0; if X>=0 then
   DECIMROUND:=XF else
   DECIMROUND:=-XF
end;
procedure CHECKLIST;
begin writeln;
   writeln('ALFA ABSCISSA ORDINATE
   POLYNOMIAL ERROR');
   for i: = 0 to 81 do
   begin AL:=0.01745329252*(i);
      XABSC:=ABSCISSA; Y:=ORDINATE;
      if XABSC<=0 then Q:=0
      else Q:=exp (EXPNT*1n(XABSC));
      YAPPR:=b[NMAX]; for j:=1 to NMAX do
      YAPPR:=b[NMAX-j]+Q*YAPR;
      writeln(i:3,XABSC:14:10,Y:14:10,YAPPR:14
      :10, (YAPPR-Y):14:10)
   end
end;
```

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```
begin write('degree of polynomial = ');
  readln(NMAX);
  write('places of decimals = '); readln(DCML);
  EXPNT:=DECIMROUND(EXPONENT);
  writeln('EXPONENT =', EXPNT:15:11);
  for i:=1 to 17 do
  begin AL:=0.07*(i-0.9); pn[i]:=0; pn1[i]:=1;
  write('*');
   x[i]:=exp(EXPNT*1n(ABSCISSA));
  f[i]:=ORDINATE
  end; for n:=0 to NMAX do
  begin write (n:2); p1:=0; x1:=0; f1:=0;
  for i:=1 to 17 do
   begin h:=sqr(pn1[i]); p1:=p1+h;
    f1:=f1+f[i]*pn1[i]; x1:=x1+x[i]*h
   end; a[n]:=f1/p1; b[n]:=a[n]; if n<NMAX
    then
   begin if n=0 then beta[1]:=0 else beta[n
    +1]:=pl/p;
    alfa[n+1]:=x1/p1; for i:=1 to 17 do
    begin pp:=(x[i]-alfa[n +1])*
     pnl[i]-beta[n+1]*pn[i];
     pn[i]:=pn1[i]; pn1[i]:=pp
    end; p:=p1
   end; gotoXY(1,whereY); C1rEo1
  end; for j:=0 to NMAX do
  begin for k:=NMAX-j-1 downto 0 do
   begin
     b[k+j]:=b[k+j]-alfa[k+1]*b[k+j+1];
    if k+j<>NMAX-1 then
     b[k+j]:=b[k+j]-beta[k+2]*b[k+j+2]
   end; b[j]:=DECIMROUND(b[j])
  end; writeln('COEFFICIENTEN');
  for j:=0 to NMAX do writeln(j:9,b[j]:16:11);
  CHECKLIST
end.
```

*the value of the exponent, referred to as EXPONENT,
*the degree of the polynomial. (For example, try 8.)
*the places of decimals, (For example, try 10.)

The full text of the program is shown in Table 1.

The program is assumed to be used for gear trigonometric functions, expressed in the pressure angle. The 17 target points for the polynomial cover a range for the pressure angle from 0.4° to 64.6°. The polynomial may have the best accuracy near these target points, but the final check should examine the less accurate points. Therefore, the final check uses pressure angles in whole numbers, and the polynomial target points lay between them.

The polynomial program yields the values of the exponent and the coefficients to be applied in the conversion procedure. Table 2 presents an example of such a procedure or function. The function name is written with a letter K instead of the letter c in secant to emphasize that it is a function of the involute INV instead of a function of an angle.

Similarly, the cosinus function and the angle itself can be computed with conversion procedures. (See Tables 3 and 4.)

Each coefficient being determined in the polynomial program depends on previously calculated ones. To be sure that the output of coefficients of the polynomial program to be written in the conversion procedure, with a certain number of decimal places, is the key to an accurate result, the coefficients are rounded off in the polynomial program immedi-

Table 2 – Conversion Procedure SEKANS(INV)

```
function SEKANS(INV:real):real;
var Q:real;
begin if INV>0 then
  begin Q:=exp(0.6666666667*In(INV));
   SEKANS:=1.0000000001+
    Q*( 1.0400419016 + Q*(0.3245063564+
    Q*(-0.0032156523 + Q*(-0.0088935917+
    Q*(0.0030544551 + Q*(-0.0002575881+
    Q*(-0.0001768974 + Q*(0.0000558091))))))))
  end else SEKANS:=1
end;
```

SEKANS is accurate to 1 unit
of the tenth decimal for a pressure angle up to 46°,
of the ninth decimal for a pressure angle up to 61°.

Table 3 – Conversion Procedure KOSINUS(INV)

```
Q:=exp(0.6666666667*In(INV));
KOSINUS:=0.9999999997+
  Q*(-1.0400418300 + Q*( 0.7571779479+
  Q*(-0.4467429639 + Q*( 0.2242268843+
  Q*(-0.0972737495 + Q*( 0.0356321395+
  Q*(-0.0098523283 + Q*(0.0014817285))))))));
```

the secant function 1/KOSINUS is accurate to 1 unit
of the ninth decimal for a pressure angle up to 42°, and
of the eighth decimal for a pressure angle up to 57°.

Table 4 – Conversion Procedure ALFA(INV)

```
Q:=exp(0.3333333333*In(INV));
ALFA:=-0.0000000278+
  Q*( 1.4422443987 + Q*(0.0001360164+
  Q*(-0.4014067126 + Q*( 0.0073800066+
  Q*( 0.0852577851 + Q*( 0.0366671471+
  Q*(-0.0498112377 + Q*( 0.0118002431))))))));
```

ALFA in radian is accurate to 4 units of the eighth
decimal (0.000002°) for a pressure angle up to 61°.

The secant function 1/cos(ALFA) is accurate to 1
unit
of the eighth decimal for a pressure angle up to 29°,
to 1 unit
of the seventh decimal for a pressure angle up to
61°.

ately after their coming into being. Special attention has to be paid to the accuracy of the exponent. If the rounding off in the conversion procedure differs from that in the polynomial program, then the loss of accuracy in the result may be as serious as unnecessary.

For application purposes the accuracy and the computing time are important features. The accuracy of the conversion procedure is very high. To achieve the same accuracy, an iteration procedure needs three steps up to about 40°, or four steps up to about 55°. Comparing the different functions in Tables 2, 3, and 4, shows that the direct calculation of the SEKANS is the most accurate one. The functions 1/KOSINUS and 1/cos(ALFA) are significantly less accurate, *in spite of the excellent accuracy of ALFA itself. The benefits* of accuracy go together with the advantage of time-saving, as is shown in Table 5.

Table 5 — Time in Milliseconds

given $1/\cos\alpha$, computed $\text{inv}\alpha$	
(using Equations 1 and 2)	Time 19 ms
given $\text{inv}\alpha$, computed SEKANS	
(using polynomial degree 6)	Time 33 ms
(using polynomial degree 8)	Time 36 ms
given $\text{inv}\alpha$, computed $1/\cos\alpha$	
(using iteration 2 steps)	Time 90 ms
(Using iteration 3 steps)	Time 115 ms

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