Hypoid Gears with Involute Teeth

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This paper presents the geometric design of hypoid gears with involute gear teeth. An overview of face cutting techniques prevalent in hypoid gear fabrication is presented. Next, the specification of a planar involute rack is reviewed. This rack is used to define a variable diameter cutter based upon a system of cylindroidal coordinates; thus, a cursory presentation of cylindroidal coordinates is included. A mapping transforms the planar involute rack into a variable diameter cutter using the cylindroidal coordinates. Hypoid gears are based on the envelope of this cutter. A hypoid gear set is presented based on an automotive rear axle.

Background

Cylindrical gearing is the simplest of all gear types and is used more than any other. Bevel and hypoid gear manufacturing analysis entail spatial, geometric relations, whereas spur and helical gear manufacturing analysis entail mostly planar geometric relations. Existing methods of manufacture for different gear forms are type-specific and generally unrelated. For example, the machines used to produce hypoid gears cannot be readily used or for fabricating spur cylindrical gears. The methods of manufacture associated with bevel and hypoid gears do not allow these gears to be treated with the same type of geometric considerations that currently exist for cylindrical gears. To illustrate, spur cylindrical gears are helical gears with a zero helix angle; both gear types can be produced using the same machine. But spur hyperboloidal gears cannot be readily produced using existing fabrication techniques for spiral hyperboloidal gears. The majority of hypoid and bevel gear manufacture today is the focus of The Gleason Corporation and Klingelnberg-Oerlikon. The following three companies provide the machines and machine tools necessary for the production of hypoid gears:

- The Gleason Works (*www.gleason.com*)
  Klingelnberg-Oerlikon (*www.klingelnberg.com*)
- Yutaka Seimitsu Kogyo LTD (*http://www.yutaka.co.jp/Y_hp6/default2.htm*)

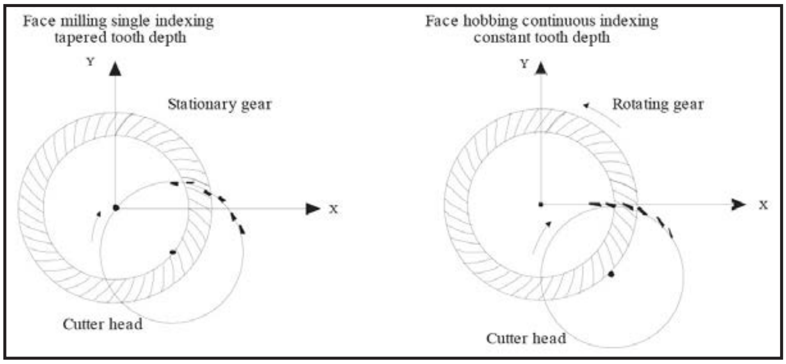

Figure 1 **Face cutting.**

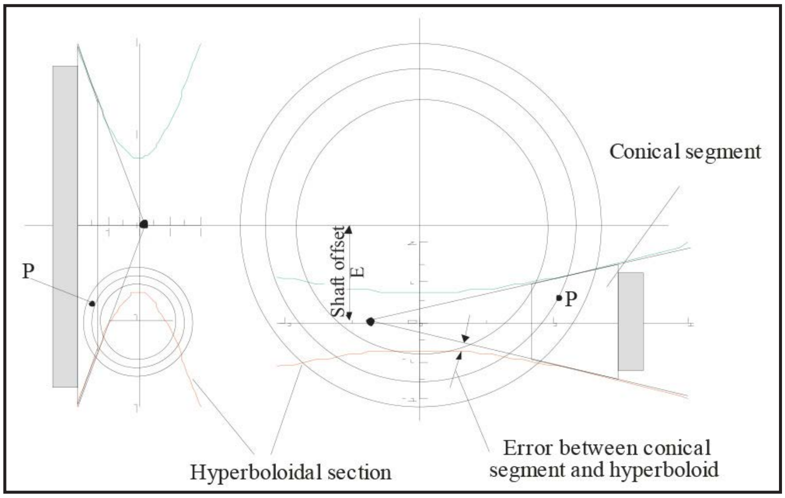

Figure 2 **Conical segments for hypoid gears.**

Depicted in Figure 1 are circular face cutters used today for fabricating spiral bevel and hypoid gear elements. Certain limitations of existing crossed-axis gear technology can be realized by focusing on Figure 2. The theoretical or ideal shape of these crossed-axis gears is the "hour-glass" — or hyperboloidal — shape shown. Current design and manufacturing techniques approximate a portion of the hour-glass shape by a conical segment, as shown. This approximation results in the following restrictions:

- Face width
- Minimum number of teeth
- Spiral angle
- Pressure angle

These restrictions in turn limit candidate gear designs. Face cutting further places restrictions on the above limitations, together with the gear ratio. An overview of face cutting methods for hypoid gear design and manufacture is provided by Shtipelman (Ref. 1); Stadtfeld (Ref. 2); Wu and Lou (Ref. 3); Wang and Ghosh (Ref. 4); and Litvin and Fuentes (Ref. 5). Radzevich (Ref. 6) and Kapelevich (Ref. 7) provide updated approaches for gear design and manufacture. Preliminary investigations into the "ideal" kinematic geometry of spatial gearing have been recognized by Xiao and Yang (Ref. 8); Figliolini et al. (Ref. 9); Hestenes (Ref. 10); and Ito and Takahashi (Ref. 11). Grill (Ref. 12) uses an "equation of meshing" to establish a relation between the curvature of one body to that of another body and applies his results in the context of gearing. Baozhen et al use Lie Algebra for a coordinate-free approach akin to screw

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theory (Ref. 13). Phillips (Ref. 14) proposes a qualitative approach for point contact of hypoid "involute" teeth. A hyperboloidal cutter was proposed as part of a unified methodology for the analysis, synthesis, and manufacture of generalized gear pairs (Ref. 15).

The manufacture of generalized gear elements is proposed by introducing a hyperboloidal or variable diameter cutter to mesh with a desired gear. An illustration of a hyperboloidal hob cutter and hypoid gear/work piece is depicted in Figure 3. The desired gear depends upon the cutter geometry along with its position and orientation relative to the gear. Two toothed bodies in mesh where the sum $\psi_{pi}+\psi_{po}$ of the spiral angles is non-zero is established to determine the cutter's position and orientation relative to the gear element.

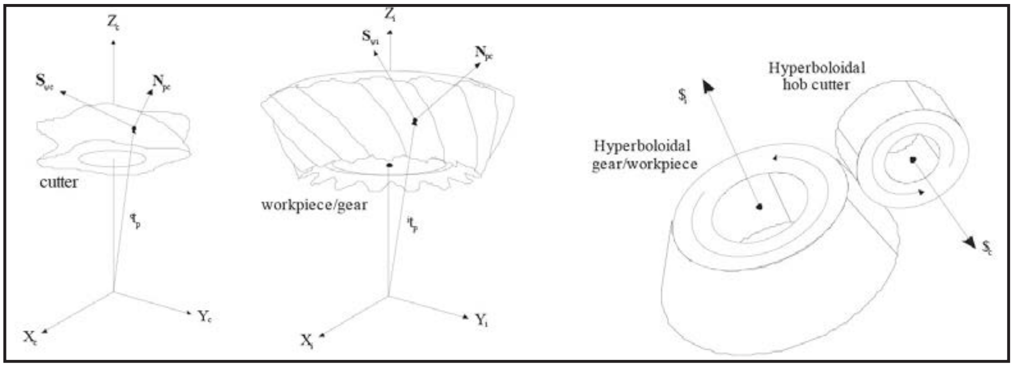

Figure 3 Cutter and gear elements.

The most common occurrence where $\psi_{pi}+\psi_{pc}\neq 0$ is for crossed cylindrical gears. The included angles α_{pi} and α_{pc} for cylindrical toothed bodies are zero, and the angle between the two axes $\$_i$ and $\$_c$ reduces to $\psi_{pi}+\psi_{pc}=\Sigma$. Meshing conditions where $\psi_{pi}+\psi_{pc}\neq 0$ and $\alpha_{pi}+\alpha_{pc}\neq 0$ are defined as *crossed hyperboloidal gears*. The I/O relationship for the meshing or generating cylindroid between two crossed hyperboloidal gears in mesh is identified by an "*s*" subscript and is uniquely defined as the *swivel I/O relationship* g_s. Generating conditions are determined using g_s, the *swivel center distance* E_s, and the *swivel shaft angle* Σ_s.

Rack Coordinates

Introducing the rack as an intermediate step for defining a candidate cutter is based on its simplicity and usefulness in transforming rotary motion into linear motion. Rack coordinates used to parameterize a gear tooth repeat each pitch P_d, thus, it is necessary to parameterize candidate rack tooth profiles for one pitch. The "*r*" subscript is used to designate that the indicated variable is in regards to the rack. If the teeth are symmetric about a line through the center of the tooth, then candidate tooth profiles need to be specified only for one-half of the pitch P_d. The Cartesian coordinates (x_r, y_r) for the rack shown in Figure 4 are divided into three regions — 1) crest; 2) active region; and 3) fillet.

This is achieved by specifying the coordinates (x_r, y_r) for the active region according to a particular application. For example, if zero errors in the I/O relationship *g* must be achieved for small changes in center distance *E*, then, as anticipated, the active profile becomes a straight line. Subsequently, the crest is determined by the "optimal" fillet of the generated gear blank. This occurs because the crest of the cutter determines the fillet of the generated blank. The fillet of the cutter is determined such that the crest of one gear pair does not interfere with the fillet of the mating gear.

The diametral pitch P_d is defined as the number of teeth per inch of pitch diameter for spur circular gears. For two toothed wheels in mesh, this leads to:

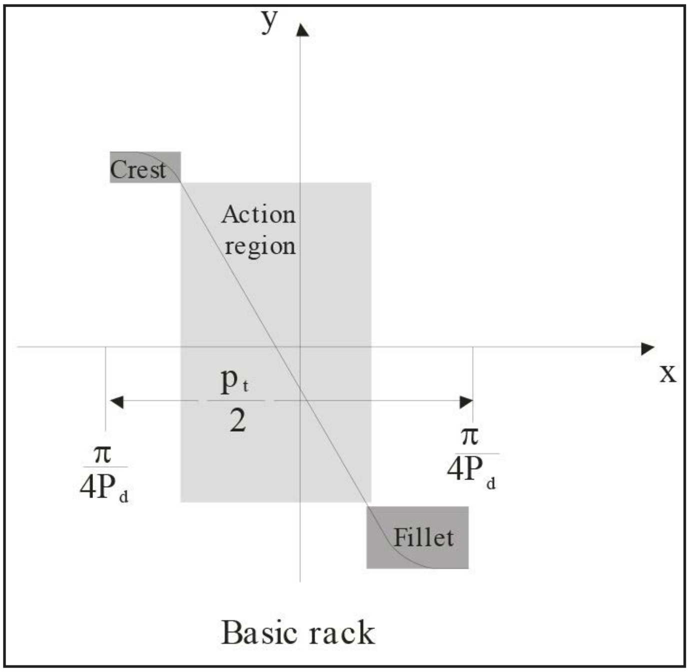

Figure 4 The rack.

$$P_d=\frac{N_i}{2u_{pi}}=\frac{N_o}{2u_{po}} \tag{1}$$

Where

N_i = Number of teeth on the input
u_{pi} = Pitch radius of the input
N_o = Number of teeth on the output
u_{po} = Pitch radius of the output

Recognizing that $u_{pi}+u_{po}=E$, where *E* is the distance between the two axes of rotation, the diametral pitch P_d is expressed:

$$P_d=\frac{N_i+N_o}{2E} \tag{2}$$

The module m_d is used in the metric system, where:

$$m_d=\frac{d_i}{N_i}=\frac{1}{P_d} \tag{3}$$

Such an expression for the tooth size is ingenious and is used to specify the addendum and dedendum height. The distance p_n between adjacent teeth can also be expressed in terms of diametral pitch P_d. Two gears in mesh must have the same p_n or normal pitch. In turn, this normal pitch can be resolved into a transverse pitch p_t and an axial pitch p_a. At this point it is convenient to temporarily abandon this terminology and introduce the distance between adjacent teeth as the *circular pitch* c_p with no indication as to whether it is the transverse, axial, or normal pitch.

A mapping is used to transform the rack coordinates (x_r, y_r) to polar gear tooth coordinates (u_c, v_c). This transformation can be envisioned as wrapping a rack onto a pitch circle with the desired pitch radius u_p. This transformation is the envelope of the rack as it meshes with a circle of radius u_p. Depicted in Figure 5 is a rack being wrapped onto a pitch circle with radius u_p.

Cylindroidal Coordinates

A system of curvilinear coordinates is used to parameterize the kinematic geometry of motion transmission between skew axes. These curvilinear coordinates are based upon the cylindroid determined by the two axes of rotation, $\$_i$ and $\$_o$, and are referred to as cylindroidal coordinates. Cylindroidal coordinates consist of families of pitch, transverse, and axial surfaces. Pitch surfaces are specified in terms of the axes of rotation $\$_i$ and $\$_o$. $\$_i$ is the input axis (pinion) of rotation and $\$_o$ is the output (ring) axis of rotation. Pitch surfaces are a family of ruled surfaces, and axodes are the unique pitch surfaces that depend upon a particular I/O relationship. For this reason, the pitch surfaces are referred to as the reference pitch surfaces.

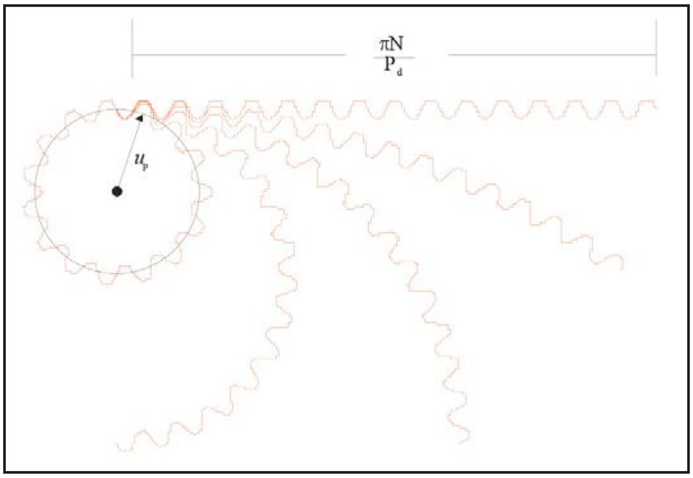

Figure 5 Transforming or "wrapping" the rack onto the desired pitch circle.

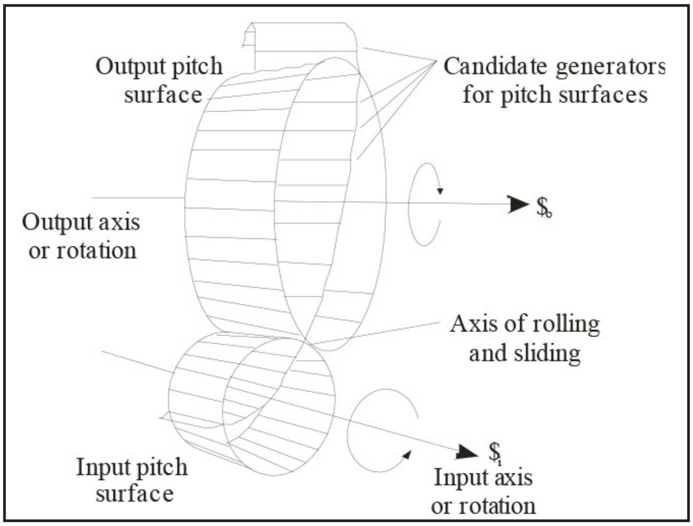

Figure 6 Two friction wheels for motion transmission between skew axes.

A system of curvilinear coordinates (u, v, w) is used to describe spiral bevel and hypoid gears. The coordinates (u, v, w) used to parameterize these families of pitch, transverse, and axial surfaces are formulated using the cylindroid defined by the input and output axes of rotation. A design methodology for spatial gearing analogous to cylindrical gearing begins with the equivalence of friction cylinders. Figure 6 shows two such wheels along with candidate generators. The I/O relationship g defines which generator of the cylindroid is used to parameterize the input and output friction wheels. These generalized friction surfaces are two ruled surfaces determined by the instantaneous generator. The transmission of motion between the two generally disposed axes $\$_i$ and $\$_o$ via two friction surfaces requires knowledge of the instantaneous generator. The location of the instantaneous generator relative to the two axes $\$_i$ and $\$_o$ depends upon:

- Distance E along the common perpendicular to axes of rotation $\$_i$ and $\$_o$
- Angle Σ between axes of rotation $\$_i$ and $\$_o$
- Magnitude of the I/O relationship g

Motion transmission between the two skew axes $\$_i$ and $\$_o$ results in a combination of an angular displacement about the instantaneous generator and a linear displacement along the instantaneous generator. The ratio h of linear displacement to that of the angular displacement is the *pitch* associated with the instantaneous generator. The pitch h_{isa} associated with the instantaneous generator is the *instantaneous screw axis,* or *ISA*.

A transverse surface is an infinitesimally thin surface used to parameterize conjugate surfaces for direct contact between two axes. Candidate generators for the reference pitch surface are determined by the generators of the cylindroid $(\$_i; \$_o)$. Given g, each position angular v_i and axial position w_i define a unique point p in space. Allowing g to vary from $-\infty$ to ∞, the point p traces a curve in space. Another value of the input position v_i defines the same cylindroid. There is an angular displacement between these two cylindroids. It is this two-parameter loci of points p that compose the transverse surface. The Cartesian coordinates r for the single point p on the generator $\$_{ai}$ are:

$$r = u_i \hat{\imath} + w_i \sin\alpha_i \hat{\jmath} + w_i \cos\alpha_i \hat{k} \tag{4}$$

Rotating the above curve r about the z_i-axis an amount v_i leads to:

$$r = \begin{bmatrix} \cos v_i & \sin v_i & 0 \\ -\sin v_i & \cos v_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ -w_i \sin\alpha_i \\ w_i \cos\alpha_i \end{bmatrix} \tag{5}$$

Where

u radius of hyperboloidal pitch surface (at throat)
v angular position of generator on pitch surface
w axial position along generator of pitch surface
α angle between generator and central axis of pitch surface

The *axial surface* provides the relationship between successive transverse surfaces. For each value of v_i, the axial surface is the loci of generators determined by g, where $-\infty < g < \infty$. The curves defined by holding two of the three parameters u, v, and w constant are coordinate curves. Two parameters used to define a surface are the curvilinear coordinates of that surface: the pitch surface by v_i and w_i ($u_i = constant$), the transverse surface by u_i and v_i ($w_i = constant$), and the axial surface by u_i and w_i ($v_i = constant$). Depicted in Figure 7 are the pitch, transverse, and axial surfaces determined using cylindroidal coordinates (u_i, v_i, w_i). Three surfaces are used to describe the geometry of gear elements.

The curvilinear coordinates (u_c, v_c, w_c) used to parameterize the proposed cutters are defined by introducing a cutter-cylindroid ($\$_{ci}; \$_{co}$). This enables cutters to be designed in pairs analogous to the design of gear pairs where two cutters are proposed for the fabrication of spiral toothed bodies. One feature of the cutter cylindroid is that expressions involving the cutters are obtained by simply changing the trailing subscripts in existing expressions involving the input gear from "*i*" to "*c*". In order to minimize the notation necessary to distinguish the input cutter from the output cutter, only a "*c*" subscript is used with no indication as to whether it is the input cutter or the output cutter. Implicit in the cutter designation will be an "*o*" subscript when describing the input gear. Likewise, when describing the output gear body, it will be assumed that associated with the cutter is an "*i*" subscript to identify that it designates the input cutter. The above reasoning is that two toothed bodies in mesh involve an input and an output body. The three possibilities being:

- Input gear body and an output gear body
- Input gear body and an output cutter
- Input cutter and an output gear body

The two twist axes $\$_{ci}$ and $\$_{co}$ are the two screws of zero pitch on the cutter cylindroid ($\$_{ci}; \$_{co}$). The generators $\$_{pc}$ are determined by also introducing a cutter I/O relationship g_c. Expressions for the radius u_{ac} and the angle α_{ac} are identical to those for u_{ai} and α_{ai}, except E, Σ, and g are replaced by E_c, Σ_c, and g_c, respectively.

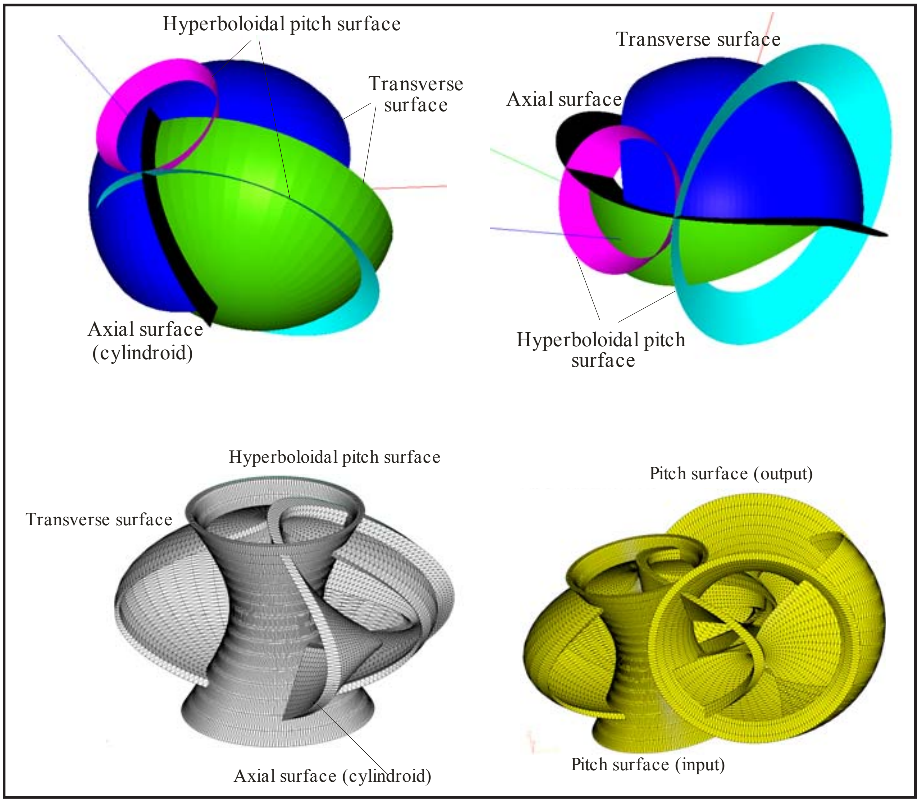

Figure 7 Pitch, transverse, and axial surface for uniform motion transmission.

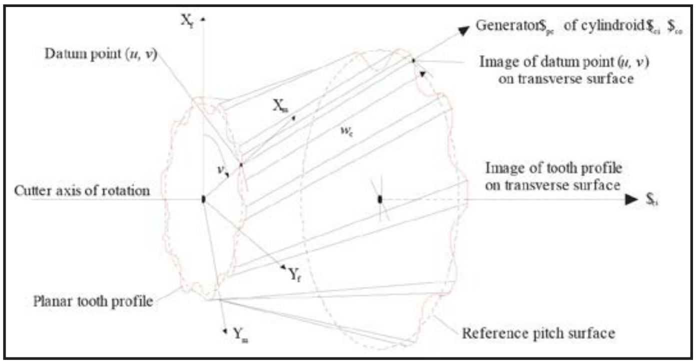

Figure 8 Mapping of planar gear profile onto transverse surface.

Hyperboloidal Cutter Coordinates

General hyperboloidal cutter elements are defined by introducing a mapping within a system of cylindroidal coordinates. The purpose of this mapping is to utilize knowledge of conjugate curves for motion transmission between parallel axes and apply it to conjugate surfaces for motion transmission between skew axes. A visual representation of this mapping is shown in Figure 8. There exists a single generator within a system of curvilinear coordinates as part of the cylindroid ($\$_i; \$_o$) that is coincident with each point (u, v). For an arbitrary axial position w_c along this generator, a transverse surface exists. Each value (u, v) defines a different generator. The distance w_c along each of these generators from (u, v) to a single transverse surface is constant. It is the image of these datum points (u, v) upon a given transverse surface that defines the mapping. This mapping is valid for any type of cutter tooth profile (viz., involute, cyclodial, circular-arc, and convuloid).

The planar coordinates (u, v) used to define conjugate curves are polar coordinates where v is an angular position about the "z-axis" and u is the corresponding radius. Use of coordinates (u, v) to specify conjugate curves in the plane are fashioned such that conjugate surfaces in space are obtained using the cylindroidal coordinates (u_c, v_c, w_c). This is achieved by assigning a value to the axial position w_c and defining $u_c \equiv u_c$ and $v_c \equiv v$. Cutter coordinates must be "scaled" to satisfy the appropriate transverse pitches. Such scaling is illustrated in Figure 9 and is obtained by recognizing that the virtual length of the striction curve s_{pc} is the component of its length perpendicular to the tooth. This scaling is performed prior to the "wrapping" of the rack onto the circular disk depicted in Figure 3 and depends on the diametral pitch. The diametral pitch P_d used to parameterize the cutter teeth depends on the size or radius of the input and output cutter. The x-scaling or stretch along the x-axis is shown in Figure 9 and depends on the cone angle α_{pc}; thus, for an arbitrary angle v_c, the corresponding parameter x_r used to evaluate the expressions for the tooth profile becomes:

$$x_c = \chi_x u_{pc} v_c \tag{6}$$

Where

$$\chi_x = \cos\alpha_{pc} \cos\gamma_{pc}$$

The angle $a_{pc} = \gamma_{pc}$ at the throat (i.e., $w_c = 0$). It is the diametral pitch at the throat that is used to specify the pitch of the cutter profiles. The cutter is expressed using the Cartesian coordinates (x_c, y_c, z_c) as follows:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} \cos v_c & \sin v_c & 0 \\ -\sin v_c & \cos v_c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_c \\ -w_c \sin v_c \\ w_c \cos v_c \end{bmatrix} \tag{7}$$

The image of the coordinates (x_c, y_c, z_c) upon the transverse surfaces must account for the cutter spiral. Consequently, a transverse angular displacement $\Delta v_{\psi c}$ is superimposed on the mapping as follows:

$$\Gamma_c = \begin{bmatrix} \cos\Delta v_{\psi c} & \sin\Delta v_{\psi c} & 0 \\ -\sin\Delta v_{\psi c} & \cos\Delta v_{\psi c} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{8}$$

The cutter spiral depends on the ratio between the axial displacement $\Delta w_{\psi c}$ and the angular displacement $\Delta v_{\psi c}$. The displacement $\Delta v_{\psi c}$ is based on a constant lead for a given transverse surface and the spiral angles ψ_c for each radii u_c are different. Note that the displacement $\Delta v_{\psi c}$ is based on the lead for the reference pitch surface and the spiral angles ψ_c change for each radius u_c.

Illustrative Example

This example presents a spiral hypoid gear set for motion transmission between skew axes using *Delgear* software (Ref. 16). The shaft angle is 90° and the shaft offset is 25 mm. The speed ratio 3.27; 11 teeth on the pinion and 36 teeth on the ring gear. The face width is 35 mm, the axial contact ratio is 3.0 and the nominal spiral angle is 61°. The tooth profile is a standard invo-

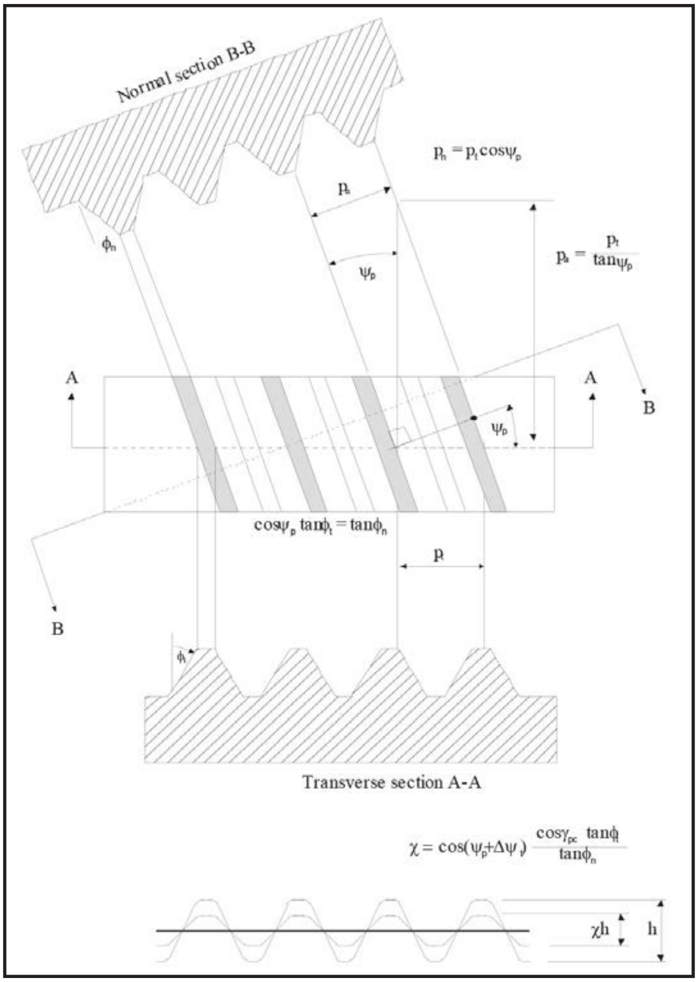

Figure 9 Scaling of tooth profile on cutter element.

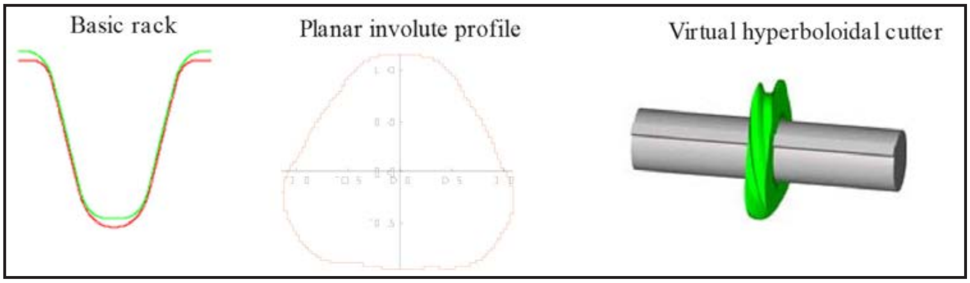

Figure 10 Rack, transverse profile, hyperboloidal cutter.

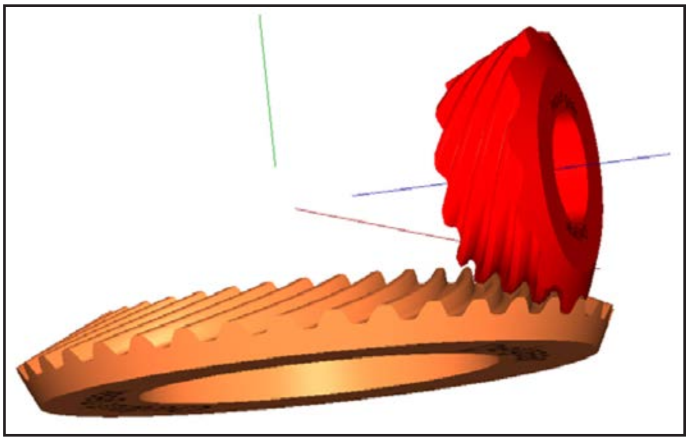

Figure 11 Input and output gears with involute teeth.

lute tooth profile. The normal pressure angle is 20°, the transverse contact ratio is 1.25, the addendum constant is 1.0 and the dedendum constant is 1.2. The variable diameter cutter has three teeth and the nominal lead angle is 10°. Figure 10 shows the rack tooth, a transverse segment of the cutter, and a virtual model of the cutter. The gear pair is depicted in mesh in Figure 11.

Summary

Demonstrated is the specification of involute gear teeth on hypoid gears. This process involves the specification of a classical involute rack, a mapping that transforms this rack to a planar circular profile. A system of cylindroidal coordinates is used to define hyperboloidal cutters. Another transformation is used to map the planar circular profile to a hyperboloidal cutter with suitable geometry for specifying general spiral bevel and hypoid gear pairs. An example of an automotive rear differential gear set is presented to illustrate the process.

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