Minimal Tooth Number of Flexspline in Harmonic Gear Drive with External Wave Generator

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Wave generators are located inside of flexsplines in most harmonic gear drive devices.

Because the teeth on the wheel rim of the flexspline are distributed radially, there is a bigger stress concentration on the tooth root of the flexspline meshing with a circular spline, where a fatigue fracture is more likely to occur under the alternating force exerted by the wave generator.

Our solution to this problem is to place the wave generator outside of the flexspline, which is a scheme named harmonic gear drive (HGD) with external wave generator (EWG). A formula to calculate the minimal tooth number of a flexspline that satisfies the requirement in fatigue strength for the flexspline is derived in this paper.

Nomenclature

Z_R Tooth number of flexspline
Z_G Tooth number of circular spline
m Module of flexspline and circular spline (mm)
r_m Radius of middle circle in flexspline (mm)
d_m Diameter of middle circle in flexspline (mm)
d_R Diameter of reference circle on flexspline before its deformation (mm)
d_G Diameter of reference circle on circular spline (mm)
h_{fR} Dedendum of flexspline tooth (mm)
h^*_{fr} Coefficient dedendum for flexspline
d Wall thickness of the flexspline's rim (mm)
E Material elastic modulus of the flexspline (MPa)
I Moment of inertia for cross section in the flexspline to mid-axis (mm^4)
B Axial width of the flexspline (mm)
W_0 Radial maximum deformation of the flexspline (mm)
b Length of short semi-axis of elliptic mid-layer (mm)
a Length of major semi-axis of elliptic mid-layer (mm)
n_s Safety coefficient under bending normal stress
n_τ Safety coefficient under torsional shear stress
$[n]$ Permissible safety coefficient
σ_{-1} Fatigue strength limit in bending for flexspline's material (MPa)
K_σ Factor of stress concentration in the gear's root influence of average stress on fatigue strength
ψ_σ Influence of average stress on fatigue strength

Introduction

It is via the elastic deformation of the flexspline that a harmonic gear drive (HGD) transfers motion and power. Unlike a conventional gear drive, which is based on the concept of rigid bodies, HGD, operated by the elastic theory, is mainly composed of three basic components: a non-rigid "flexspline," a rigid "circular spline" and an elliptical "wave generator." The wave generator produces and controls the deformation of the flexspline. It is called the harmonic drive since the wave form presented in the flexspline is a simple harmonic wave which is symmetric by and large (Refs. 1 and 2).

Harmonic gear drives have advantages that other drives do not, such as simpler structure; higher drive ratio with a wide range of variation, which is capable of producing speed ratios up to 300 in a single stage (Ref. 2); larger load-carrying capacity resulting from more teeth meshing and supporting load simultaneously; tooth contacts of about 10-30% of the total tooth number (Ref. 3); and smooth transmission with no impact between gears. Because of these advantages, HGD has promising applications in fields such as aerospace, machine tools, instruments, electronic equipment, mining equipment, transport and communication facilities, hoisting machinery, petrochemical machinery, textile machines, agricultural machinery, medical apparatus and instruments (Refs. 1–3).

A flexspline in operation undergoes alternating stress; its life determines the service life of the entire harmonic gear device. The wave generators are mostly located inside of flexsplines in current HGD devices. Because the teeth of the flexspline are distributed radially on the gear rim, the mouth of the tooth socket in the flexspline meshing with the circular spline will open wider under the action of wave generator. Higher stress concentration is caused in the tooth root of the flexspline, where a fatigue crack occurs easily under the periodically alternating force of the wave generator.

If the wave generator is set outside of the flexspline — known as an HGD/EWG (harmonic gear drive with external wave generator) — the teeth on the rim of the flexspline are distributed in a form of convergence on this occasion. The stress concentration in the tooth root of the flexspline is lightened, thus enhancing the strength of the flexspline and, in turn, prolonging the service life of the device.

The structure size of two key gears — the flexspline and the circular spline — must be determined first in order to design an HGD mechanism with EWG. Unfortunately, HGD-with-EWG is rarely discussed in mechanical design literature, nor mentioned in reference material.

In an attempt to address the problem, this paper presents a formula to calculate the minimal tooth number of flexspline in double-wave HGD-with-EWG. The formula is rooted in the concept of a mid-layer in the flexspline and the analysis of the radial deformation of the mid-layer, section stress and fatigue strength of the flexspline under the action of EWG. The tooth number of the circular spline can then be settled; if its tooth number is determined, a theoretical foundation is thus laid for structural design and strength analysis on a harmonic gear device with EWG.

Principle of HGD-with-EWG

An HGD mechanism with EWG (Ref. 4) is also composed of three basic parts: 1) flexspline; 2) circular spline; and 3) wave generator (Fig.1). The circular spline is an external gear; the flexspline is a thin-wall gear ring with internal teeth, and its tooth number is slightly more than that of the circular spline. Most harmonic gear devices are of double waves, and, in this case, the tooth number of the flexspline is usually two more than that of the circular spline.

Initially, the profile of the flexspline is circular in shape (Figs. 1 and 2) before the flexspline is compressed by the EWG. The inner side diameter of the wave generator is slightly less than the outside diameter (equal to $2r_m$) of the flexspline before it is deformed. The flexspline is squeezed into an oval under the action of EWG, which contacts with the flexspline. Teeth near both ends of the minor axis of the ellipse for the flexspline mesh completely with the teeth in the circular spline, and the teeth near the major axis of the ellipse separate completely from those in the circular spline; teeth on other segments are in transition states of coming into or out of mesh gradually.

The circular pitches for the two gears are the same. In the pitch circle of the circular spline, the meshing process for teeth of the flexspline and circular spline looks like two pure rolling (no sliding) rings. The arc length rolled in the pitch circle must be equal at any moment for two gears. The deformation position on the flexspline is ever changing; EWG turns continuously as the teeth of the two gears are in mesh, and then out of mesh; in turn, the engagement transmission is thereby realized.

Radial Deformation of a Flexspline

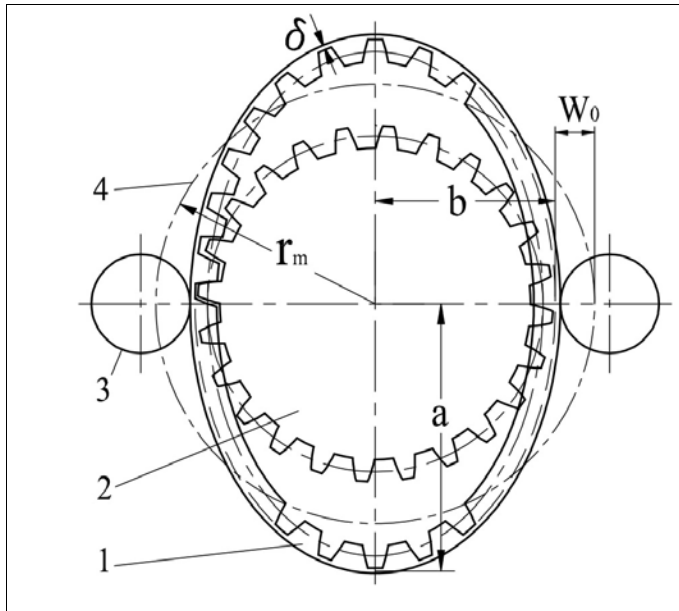
The concept of mid-layer in flexspline. Elastic deformation of the flexspline emerges periodically under the action of EWG. The deformation of the flexspline is usually represented by the deformation of the mid-layer. The so-called mid-layer, a hypothesis in material mechanics (Ref. 5), refers to the middle layer in the interior of the flexspline's rim, which is neither elongated nor shortened while the flexspline is undergoing bending deformation. The middle face through the rim thickness of the ring-flexspline is usually considered as the mid-layer because the degree of bending deformation for the flexspline is less (Ref. 6). The line intersection of mid-layer and cross-section is called the neutral axis; i.e. — a line on the cross-section connecting every point where its normal stress is zero. The elliptic mid-layer in the flexspline will rotate as EWG turns.

The mid-layer of the flexspline before it is deformed is a circle, or middle circle. If the flexspline is a standard gear (no addendum modification), the radius of the middle circle in the flexspline (Figs. 1 and Fig. 2) is determined by the equation:

$$r_m = 0.5d_R + h_{fR} + 0.5\delta \quad (1)$$

where:

the diameter of the reference circle on the flexspline before its deformation $d_R = mZ_R$, the dedendum of flexspline tooth $h_{fR} = mh^*_{fR}$, coefficient of dedendum $h^*_{fR} = 1.35$ (Ref. 7) for the normal tooth of the flexspline with a pressure angle of 20 on the reference circle according to the standard of involute tooth profile in harmonic gear universally.



1- flexspline 2- circular spline 3- wave generator 4-middle circle

Figure 1 Generalized drawing of HGD with EWG.

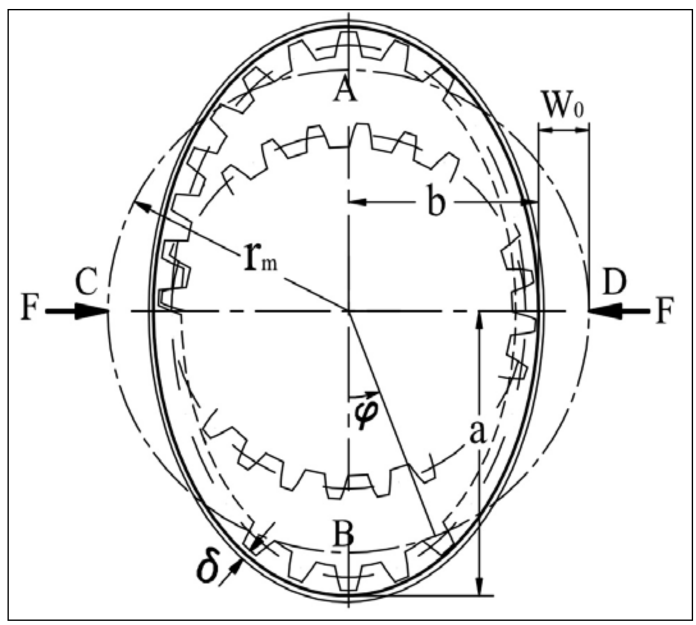


Figure 2 Shape change of mid-layer in flexspline — from circle to ellipse.

Bending moment on section in flexspline. The flexspline in Figure 1 — a cylindrical shell with teeth inside — is an orthotropic anisotropic shell (for the flange or rim of the flexspline) in structure. There exist structural nonlinearities, since the teeth on the rim are equivalent to longitudinal reinforcing ribs, and geometric nonlinearities of large deflection because the ratio of radial maximum deformation to rim's wall-thickness of the flexspline is bigger than 0.2. It is therefore difficult to obtain an exact solution in internal force, deformation and stress distribution on the flexspline's rim under the action of EWG.

Some simplification is necessary in order to study the internal force and strength of the flexspline. Influence of a flexspline's teeth could be left out first, so that the flexspline is simplified as a smooth, thin-walled cylindrical shell with a rim thickness δ, since it is far less than r_m, the radius of the middle circle (Ref. 8). After obtaining the deformation and stress of the flexspline shell using the shell theory in elastic-plastic mechanics, the influence of teeth on stress in smooth shell is then considered and reflected in the calculation formula as a tooth influence coefficient (Ref. 9).

Assuming that the EWG exerts concentrated forces on the flexspline, a mid-layer ring of flexspline which bears a pair of equal pressures *F* in opposite directions is taken as a research model. As shown in Figure 3a, it is a statically indeterminate structure (Ref. 5).

Just one-quarter of the ring-flexspline (Fig. 3b) need be studied, considering the symmetry of the ring's geometry and load. There is only axial force and moment on section B because of the symmetry. The axial force is equal to 0.5 *F* using equilibrium condition, and bending moment *Mx* is a redundant constraint to be solved. Since the turn-angles on sections B and D are zeroes, section D could be regarded as a fixed end, and the zero turn-angle on section B is taken as deformation coordination condition. Thus a force-method based canonical equation (Ref. 5) can be written (the derivation process is omitted here due to limited length of this article), redundant constraint (bending moment) $Mx = F_{rm}$ (0.5-1/π) is then gotten, and from which the bending moment on any section for the ring-flexspline is determined by the formula:

$$Mx = F_{rm} (0.5 \cos(\varphi) - 1/\pi) \tag{2}$$

where:
F is the extruding force pressed on the ring-flexspline by EWG; φreflects the position of the ring section starting from vertical section B, φ = 0 ~ 90, and the counterclockwise angle is considered positive.

Radial deformation of mid-layer in the flexspline. Radial deformation in major and short axes of elliptic mid-layer on the flexspline has the greatest influence on meshing quality between the flexspline and the circular spline, so the fluctuations in length must be analyzed for vertical diameter AB and horizontal diameter *CD* on the mid-layer (Figs. 2 and 3).

The length of horizontal diameter *CD* on the ring-flexspline will be shortened under a pair of extruding forces *F*. From Equation 2 we apply Moore's theorem (Ref. 5) and integrate along the entire circle of the ring, the reduction of diameter *CD* that is the relative displacement δ_{CD} between *C* and *D* on which the forces *F* act can be obtained:

$$\delta_{CD} = \frac{F r_m^3 \left(\frac{\pi}{4} - \frac{2}{\pi} \right)}{EI} \tag{3}$$

where:
E is material elastic modulus of the flexspline; *I* is the moment of inertia for cross-section in the flexspline torus to mid-axis $I = B^3/12$; *B* is the axial width of the flexspline; *EI* means flexural rigidity of the torus, which reflects material's ability of resistance to bending deformation.

The length of vertical diameter AB for the ring-flexspline will surely be elongated when the length of horizontal diameter *CD* becomes shorter. According to Equation 2, applying Moore's theorem (Ref. 5) again along the entire circle of the ring, the amount of elongation δ_{AB} for diameter AB can also be obtained:

$$\delta_{AB} = \frac{F r_m^3 \left(\frac{2}{\pi} - \frac{1}{2} \right)}{EI} \tag{4}$$

It can be seen that $\delta_{CD} > \delta_{AB}$ from Equations 3 and 4 means that the maximum radial displacement among points on the mid-layer appears in the short axis of ellipse; i.e., in the direction of extruding forces.

From Figure 2, it is directly seen that the radial maximum deformation

$$W_0 = \frac{\delta_{CD}}{2} \tag{5}$$

To simplify the discussion, assume that the flexspline and circular spline are two standard gears. From the point of view for proper meshing between two gears, the reference circle on the circular spline should coincide with the reference line near the short axis of the elliptic mid-layer of the flexspline (Fig. 1). In other words, the length of short semi-axis *b* should satisfy the following relationship:

$$b = 0.5\, d_G + h_{fR} + 0.5\delta \tag{6}$$

where:
the diameter of the reference circle on a circular spline $d_G = mZ_G$. The thickness of the flexspline's rim $\delta = (0.01 \sim 0.015)\, d_R$ is generally taken (Ref. 9).

Obviously, $W_0 = r_m - b$ (Figs. 1 and 2). We obtain $r_m - b = 0.5m (Z_R - Z_G)$ by subtracting Equation 6 from Equation 1. Parameters of W_0, *m*, Z_G and Z_R must obey the following expression:

$$W_0 = 0.5 m \Delta Z \tag{7}$$

where:
D*Z* is the tooth number difference of the flexspline and the circular spline; $DZ = (Z_R - Z_G)$.

Stress on Cross-Section in the Rim of a Flexspline

The scholar M. H. Ivanov considered that HGD is one having initial pre-stress (Ref. 8). After the wave generator is assembled outside of the flexspline, the flexspline is force-deformed and subjected to preliminary load because of the extrusion effect by EWG — thus pre-stress occurs in the cross-section of the flexspline's rim.

The maximum and minimum stresses in section of flexspline. It is known (Eq. 2) that the biggest bending moment occurs on sections D of φ = 90 in the ring-flexspline (Fig. 3); its value $M_{max} = -Fr_m/\pi$ (minus sign means that this bending moment decreases the curvature of the elliptic mid-layer). The stress at the outermost end (away from the center of the middle circle) on section D is the maximum compressive stress in cross-sections of the ring-flexspline, its value $\sigma_{max} = 0.5\, M_{max}\delta/I = -0.5 F r_m \delta/(\pi I)$. The maximum compressive stress could be calculated from Equations 3 and 5:

$$\sigma_{max} = \text{-}8.5580\frac{W_0 E\delta}{d_m^2} \quad (8)$$

Here the minus sign indicates a compressive stress; the diameter of the middle circle in the flexspline is $d_m = 2r_m$.

Also from Equation 2, it is obvious that the minimal bending moment appears on sections B of $\varphi = 0$ in the ring-flexspline; its value is $M_{min} = Fr_m$ (0.5-1/π). The stress at the outermost end (away from the center of the middle circle) on section B (it is a tensile stress) $\sigma_{min} = 0.5\ M_{min}\delta/I = 0.5\ Fr_m\ (0.5\text{-}1/\pi)\ \delta/I$, which is also the minimum tensile stress of all sections in the ring-flexspline. The minimum tension stress can be calculated as follows:

$$\sigma_{min} = 4.8849\frac{W_0 E\delta}{d_m^2} \quad (9)$$

Cycle characteristic of alternating stress on section in ring-flexspline. As mentioned above, the stress at outermost end on section D (Fig. 3) in the ring-flexspline is biggest (maximum compressive stresses), and smallest on section B (that is, minimum tensile stresses). Therefore, the stress on the ring's section is an alternating stress in an asymmetric cycle, which is a reason leading to fatigue failure for the flexspline.

Combining Equations 8 and 9, and considering the influence of teeth in the flexspline on stress, will derive a general formula to calculate the average stress in sections of flexspline in one cycle.

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} = 1.8366\frac{K_\delta W_0 E\delta}{d_m^2} \quad (10)$$

where:

K_δ is a factor reflecting the influence of teeth in flexspline on stress (Ref. 9), $K_\delta = 1.05 \sim 1.10$, in most cases.

Alternating stress amplitude, namely, the difference of maximum and minimum stresses on the ring's section, is:

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} = 6.7214\frac{K_\delta W_0 E\delta}{d_m^2} \quad (11)$$

Determining Minimal Tooth Number of Flexspline

The performance of the flexspline depends on its fatigue strength. Stress on a section in a ring-flexspline is in cyclical change (an asymmetric cycle), as described. We thus must calculate the fatigue strength of the flexspline (Ref. 6).

According to Refs. 5 and 8, the safety coefficient n in fatigue strength under alternating stress of the combination of bending and twisting is:

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}} \geq [n] \quad (12)$$

where:

the permissible safety coefficient $[n] = 1.3 \sim 1.5$ (Refs. 8 and 9), as a rule.

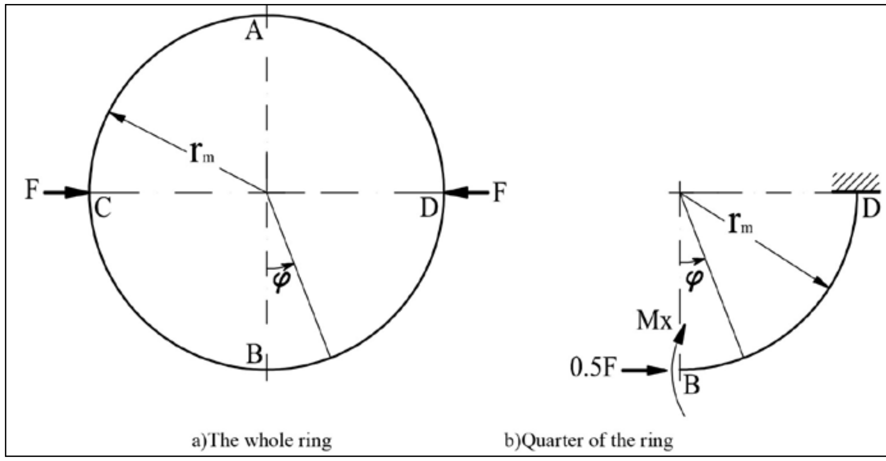


Figure 3 Force analysis on mid-layer of flex-spline.

Usually, the influence of normal bending stress is only considered in approximate calculation, since it is quite small for the influence of torsional shear stress on the fatigue strength of the flexspline (Ref. 8). That is, the following equation on safety coefficient in fatigue strength of flexspline should meet:

$$n_\sigma = \frac{\sigma_{-1}}{K_\sigma \sigma_a + \psi_\sigma \sigma_m} \geq [n] \quad (13)$$

where:

the factor of stress concentration in the gear's root $K_\sigma = 1.8 \sim 2.0$, influence factor of average stress on fatigue strength $\psi_\sigma = 0.10 \sim 0.15$ are reasonable ranges (Refs. 6 and 8).

The minimal tooth number of a flexspline satisfying the requirement in fatigue strength is then obtained from equations 1, 7, 10, 11 and 13.

In general, let's take $K_\delta = 1.07$; $K_\sigma = 1.9$, $\psi_\sigma = 0.12$; $\delta = 0.012$; $d_R = 0.012\ mZ_R$ in the recommended ranges. Substituting relevant data into Equations 1, 7, 10, 11 and 13 yields the following relationship:

$$n_\sigma = 12.2795\frac{\sigma_{-1}(Z_R + 2.668)^2}{K_\sigma \sigma_a + \psi_\sigma \sigma_m} \geq [n] \quad (14)$$

Equation 14 is the condition under which the tooth number of the flexspline should be satisfied. And ΔZ, the tooth number difference for two gears, should be integral times of the wave number for HGD (Refs. 8 and 9). Double-wave HGD with a tooth number difference of two has been widely used at present, and double-wave HGD with a tooth difference of four also has application on specific occasions; thus ΔZ could be 2 or 4 in different applications.

Tooth number Z_G of the circular spline could then be calculated using the formula:

$$Z_G = Z_R - \Delta Z \quad (15)$$

where:

if the tooth number Z_R of the flexspline and tooth number difference ΔZ of two gears are determined.

Example: Suppose we are to design a double-wave harmonic gear assembly with EWG. A steel 30CrMnSiA, quenched and nitrided, is selected as material for the flexspline; its bending fatigue limit $\sigma_{-1}$is 625 MPa; elastic modulus E is 2.1 10^5

MPa. Let $[n]=1.4$ and $\Delta Z=2$; solution of inequality (Ref. 14) is $Z_R \geq 71.2765$; it means that the tooth number of the flexspline should be at least 72, a satisfaction of condition $n_\sigma \geq [n]$ guarantees that fatigue strength of the flexspline is enough. If $\Delta Z=4$ is taken, the result from inequality 14 is $Z_R \geq 148.0406$; the tooth number of the flexspline should be more than 148. In this case, the flexspline could meet requirement for fatigue life.

Conclusions

Stress concentration near the tooth root of a flexspline is alleviated in HGD-with-EWG. The strength of the flexspline is improved and service life of the device can thus be extended, compared to HGD with built-in wave generator—today's accepted system. Therefore foundational study on HGD-with-EWG has both theoretical and practical significance.

The basic features in HGD are that the flexspline bends periodically and repeatedly under the action of the wave generator, and is in a working condition of alternating stress. HGD failure is thus mainly caused by fatigue fracture of the flexspline. Many of the benefits of HGD — such as reliability, kinematic accuracy and bearing capacity — are chiefly affected by the flexspline. Striking that delicate balance between flexspline strength and elasticity has been the focus of HGD research for a long time. For HGD-with-EWG, the flexspline is also the weakest link; its deformation and stress have critical impact upon transmission precision and the life of the entire device.

The authors analyzed the radial deformation of the mid-layer and section stress of the flexspline under the action of EWG based on the research model of a smooth cylindrical shell. It showed that the stress in the flexspline at work is an alternating one of asymmetric cycle; the stresses of cross-sections in the flexspline's rim are larger at major axis and short axis of elliptical mid-layer; those between the major axis and short axis are smaller; and the stress in the short axis is bigger than that in the major axis. The most dangerous section, vulnerable to damage, is the contact zones (sections C and D, Fig. 3) of the wave generator and the flexspline.

The strength of the flexspline is closely related to its structure parameters — especially to its number of teeth

Determination of tooth number of the flexspline must comprehensively consider such factors as the deformation, fatigue strength of the flexspline and its proper gearing with the circular spline.

The formula on the minimal tooth number of the flexspline derived in this paper provides a basis for tooth number selection and structure design on the flexspline and the circular spline in HGD-with-EWG and engineering application of this kind of device, which changes the present situation that little research is done on relationship between the strength of flexspline and its tooth number, and remedies a shortage in reference to devise HGD device with EWG.

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