Investigation of the Strength of Gear Teeth

Wilfred Lewis, Member, Engineers' Club of Philadelphia
Read October 15, 1892

To mechanical engineers, the strength of gear teeth is a question of constant recurrence, and although the problem to be solved is quite elementary in character, probably no other question could be raised upon which such a diversity of opinion exists, and in support of which such an array of rules and authorities might be quoted. In 1879, Mr. John H. Cooper, the author of a well-known work on "Belting," made an examination of the subject and found there were then in existence about forty-eight well-established rules for horsepower and working strength, sanctioned by some twenty-four authorities, and differing from each other in extreme cases of 500%. Since then, a number of new rules have been added, but as no rules have been given which take account of the actual tooth forms in common use, and as no attempt has been made to include in any formula the working stress on the material so that the engineer may see at once upon what assumption a given result is based, I trust I may be pardoned for suggesting that a further investigation is necessary or desirable.

In summing up his examination, Mr. Cooper selected the following formula from an English rule published at Newcastle-under-Lyme in 1868, and, as an expression of good general averages, it may be considered passably correct.

$$X = 2{,}0000\, p\, f$$

in which

X = breaking load of tooth in pounds
p = pitch of teeth in inches
f = face of teeth in inches

In conclusion he makes this pertinent observation: "It must be admitted that the shape of the tooth has something to do with its strength, and yet no allowance appears to have been made by the rules tabulated above for such distribution of metal, the breaking strength being based upon the pitch or thickness of the teeth at the pitch line or circle, as if the thickness at the root of the tooth were the same in all cases as it is at pitch line." Notwithstanding the fact that the necessity for considering the form of the tooth, as well as its pitch and face, was thus clearly set forth over thirteen years ago, I am not aware that anyone else has taken the trouble to do it, and, as the results to be presented have been well-tested by experience in the company with which I am connected, I believe they will be of interest and value to others.

In estimating the strength of teeth, the first question to be considered is the point or line at which the load may be applied to produce the greatest bending stress. In the rules referred to, the load is sometimes assumed to be applied at the pitch line, sometimes at the end of the tooth, and sometimes at one corner; but in good modern machinery, the agricultural type excepted, there is seldom any occasion to assume that the load is not fairly distributed across the teeth. Of course, it may be concentrated at one corner as the result of

While it is impossible to compute the load stress on an involute curved gear tooth, it is possible to compute the stress of a load imposed onto a parabola. This observation is the basis for the Lewis bending strength formula.

careless alignment or defective design, in which the shafts are too light or improperly supported, and, for a rough class of work, allowance should certainly be made for this contingency, but, in all cases where a reasonable amount of care is exercised in fitting, a full bearing across the teeth will soon be attained in service. It must be admitted, however, that on account of the inevitable yielding of shafts and bearings, even of the stiffest construction, the distribution of pressure may not be uniform under variable loads, and that the assumption of uniform pressure across the teeth is not always realized in the best practice. To what extent it is realized I shall not attempt to estimate, but in general practice, where the width of the teeth is not more than two or three times the pitch, the departure can not be regarded as serious. The conclusion is therefore reached that in first-class machinery, for which the present investigation is intended, the load can be more properly taken as well-distributed across the tooth than as concentrated at one corner. Having thus disposed of the first question, the second is, at what part of the face should the load be assumed to be carried in estimating the strength of a tooth?

Evidently the load may be carried at any point within the arc of action, and it might be argued that when a tooth is loaded at its end, there are always two teeth in gear, and that the load should be divided between them. This is theoretically true of all teeth properly formed and spaced, but it must be admitted that mechanical perfection in forming and spacing has not yet been reached, and that the slightest deviation in either respect is sufficient to concentrate the whole load at the end of a single tooth. In time this concentration may be relieved by wear, but it is not so easily corrected as unequal distribution across the teeth, and, as the present practice of cutting gears with a limited number of equidistant cutters makes it almost impossible to obtain teeth of proper shape, it is evident that the load cannot safely be assumed as concentrated at a maximum distance less than the extreme end of the tooth. In some cases, of course, the teeth will not be so severely tested, and the error in this assumption compensates in a measure for the error in the first assumption of equal distribution across the teeth. Having thus concluded that gear teeth may fairly be considered as cantilevers loaded at the end, the influence of their form upon their strength remains to be disposed of.

In interchangeable gearing, the cycloidal is probably the most common form in general use, but a strong reaction in favor of the involute system is now in progress, and I believe an involute tooth of 22-1/2° obliquity will finally supplant all other forms. There are many good reasons why such a system should be generally adopted, but it is not my purpose at present to discuss the merits and defects of different systems of interchangeable gearing, and I now propose to explain how the factors given in the table herewith were determined by graphical construction.

A number of figures were carefully drawn on a large scale, to represent the teeth cut by a complete set of equidistant cutters, making the fillets at the root as large as possible to clear an engaging rack. See Plate 1.

The addendum was $.3p$ and root $.35p$, as shown in the illustrations, and the clearance was $.02p$.

When the load is applied at the end of a tooth and normal to its face in the direction *ab*, it may be resolved into two forces, one tending to crush the tooth, and the other to break it across. The radial component, which tends to crush the tooth directly, has but a slight effect upon its strength. In material which is stronger in compression than in tension, the transverse stress due to the other component W is partially counteracted on the tension side, and the teeth are stronger by reason of their obliquity of action; but in material which is weaker in compression the reverse is the case, and, in general, it may be said that the strength of teeth will not be affected more than 10% either way by the consideration of this radial component.

It should therefore be understood that for the sake of simplicity, the factors given have been determined only with reference to the transverse stress induced by the force W, which may be regarded as the working load transmitted by the teeth. This load is applied at the point b, but it does not at once appear where the tooth is weakest, and, to determine that point, advantage is taken of the fact that any parabola in the axis *be* and tangent to *bW* at the point *b* encloses a beam of uniform strength. Of all the parabolas that may thus be drawn, one will be tangent to the tooth form, and it is evident that the point of tangency will indicate the weakest section of the tooth. In the rack tooth of 20° obliquity, this is found at once by prolonging *ca* to its intersection *g* with the center line *fb*, and laying off $bf = bg$; and in other cases, the weakest section *cd* may be found tentatively to a nice degree of accuracy in two or three trials. Having found the weakest section, the strength at that point is also determined graphically by drawing *bc* and erecting the perpendicular *ce* to intersect the center line in *e*. Then *ef* or *x* is taken to measure the strength of the various forms of teeth.

To understand the reason for this construction and the actual relation which the distance *x* bears to the strength of the tooth, it will be observed that the bending moment W*l* on the section *cd* is resisted by the fiber stress *s* into one-sixth of the face *f* times the square of the thickness *t*, or, by the well-known formula for beams, we have

$$W\,l = \frac{s\,f\,t^2}{6}, \text{ or}$$

$$W = \frac{s\,f\,t^2}{6\,l} \qquad (1)$$

But by similar triangles

$$x = \frac{t^2}{4\,l} \qquad (2)$$

and substituting this value in Equation 1 we have

$$W = s\,f\,\frac{2\,x}{3}, \qquad (3)$$

or we may write

$$W = s\,p\,f\,\frac{2\,x}{3\,p} \qquad (4)$$

The factor $\frac{2x}{3p}$ or *y* is determined by graphical construction and is given in Table 1 for convenient reference. This is multiplied by the pitch face and fiber stress allowable in any case when the working load *W* is to be determined. What fiber stress is allowable under different circumstances and conditions cannot be definitely settled at present, nor is it probable that any conclusions will be acceptable to engineers unless based upon carefully made experiments. In the article referred to, certain factors are given as applicable to certain speeds, and in the absence of any later or better light upon the subject, Table 2 has been constructed to embody in convenient form the values recommended. It cannot be doubted that slow speeds admit of higher working stresses than high speeds, but it may be questioned whether teeth running at 100 feet a minute are twice as strong as at 600 feet a minute, or four times as strong as

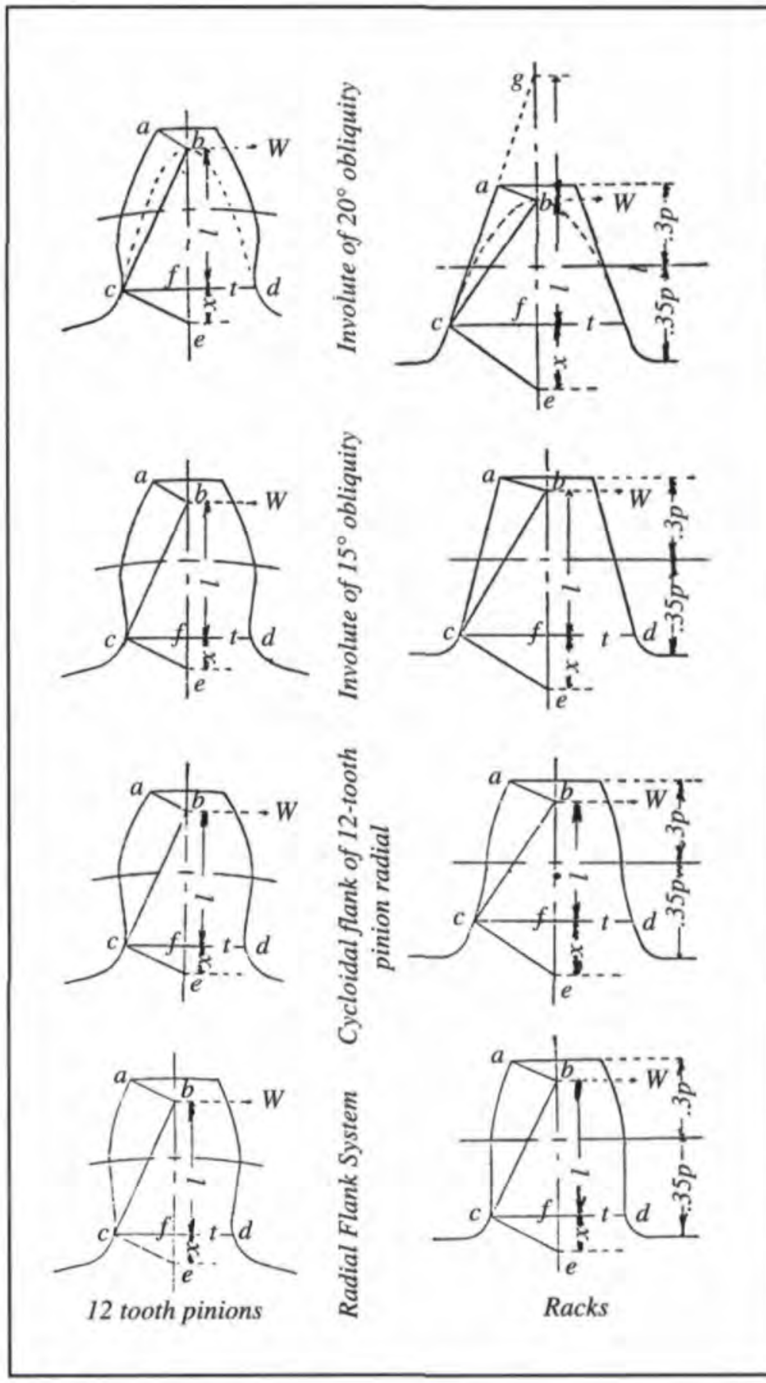

Plate I

TABLE 1

Factor for Strength, y

Number of Teeth	Involute 20° Obliquity	Involute 15° and Cycloidal	Radial Flanks
12	.078	.067	.052
13	.083	.070	.053
14	.088	.072	.054
15	.092	.075	.055
16	.094	.077	.056
17	.096	.080	.057
18	.098	.083	.058
19	.100	.087	.059
20	.102	.090	.060
21	.104	.092	.061
23	.106	.094	.062
25	.108	.097	.063
27	.111	.100	.064
30	.114	.102	.065
34	.118	.104	.066
38	.122	.107	.067
43	.126	.110	.068
50	.130	.112	.069
60	.134	.114	.070
75	.138	.116	.071
100	.142	.118	.072
150	.146	.120	.073
300	.150	.122	.074
Rack	.154	.124	.075

TABLE 2
Safe Working Stress, s, for Different Speeds

Speed of Teeth in Ft. per Minute	100 or less	200	300	600	900	1200	1800	2100
Cast Iron	8,000	6,000	4,800	4,000	3,000	2,100	2,000	1,700
Steel	20,000	15,000	12,000	10,000	7,500	6,000	5,000	1,300

the same teeth at 1,800 feet a minute. For teeth which are perfectly formed and spaced, it is difficult to see how there can be a greater difference in strength than the well-known difference occasioned by a live load or a dead load, or two to one in extreme cases. But for teeth as they actually exist, a greater difference than two to one may easily be imagined from the noise sometimes produced in running, and it should be said that this table is submitted for criticism rather than for general adoption. It is one which has given good results for a number of years in machine design, and its faults, such as they may be, are believed to be in the right direction from another point of view; for when the durability of a train of gearing is considered, it would seem that all gears in the train should have the same pitch and face, because all transmit the same power and are therefore subject to the same wear. But this argument is modified by the further consideration that equal wear does not mean equal life in a train of gearing, and a compromise between the considerations of life and strength must result in the adoption of different values for different speeds, somewhat similar to those given in Table 2.

To illustrate the use of Tables 1 and 2, let it be required to find the working strength of a 12-toothed pinion of 1" pitch, 2 1/2" face, driving a wheel of 60 teeth at 100 feet or less per minute, and let the teeth be of the 20° involute form. In the formula $W = spfy$, we have for a cast-iron pinion,

$$s = 8{,}000,\ pf = 2.5,\ \text{and}\ y = .078,$$

and multiplying these values together we have $W = 1{,}560$ pounds. For the wheel we have $y = .134$, and $W = 2{,}680$ pounds.

The cast-iron pinion is, therefore, the measure of strength, but if a steel pinion be substituted, we have $s = 20{,}000$ and $W = 3{,}900$ pounds, in which combination the wheel is the weaker, and it therefore becomes the measure of strength. In teeth of the involute and cycloidal forms, there is a marked difference between racks and pinions in working strength, while in radial flanked teeth, which are used more especially on bevel gears, the difference is not so pronounced. These teeth are to be found in the great majority of all cut bevels, because they can be more cheaply produced on milling machines and gear cutters, but the 15° involute bevel tooth, as made by the Bilgram process, is superior in accuracy of form and finish, and is often preferred for patterns and fine machinery. There are, therefore, two well-defined forms of bevel gears to be considered: and to bring the strength of bevel gearing within the scope of the present investigation, it will be necessary to understand how the variation in their pitch and radius of action is allowed for, and without going into a demonstration of the formula, its simple statement will probably be sufficient.

Referring to Plate II,

D = large diameter of bevel.

d = small diameter of bevel.

p = pitch at large diameter.

n = actual number of teeth.

N = formative number of teeth = n secant a, or the number corresponding to radius R.

f = face of bevel

y = factor depending upon shape of teeth and formative number N.

W = working load on teeth referred to in Diagram D.

Then it can be shown that

$$W = s\,p\,f\,y\,\frac{D^3 - d^3}{3\,D^2\,(D - d)} \qquad (5)$$

To illustrate the use of this formula, let it be required to find the working strength of a pair of cast-iron miter gears of 50 teeth, 2" inch pitch, 5" face, at 120 revolutions per minute.

In this case, $a = 45°$, $D = 31.8$, $d = 24.8$, and secant $a = 1.4$, $N = 50 \times 1.4 = 70$, for which $y = .071$. The speed of the teeth is 1,000 feet per minute, for which, by interpolation in the table, $s = 2{,}800$, and the formula becomes, by substituting these values.

$$W = 2{,}800 \times 2 \times 5 \times .071 \times \frac{31.8^3 - 24.8^3}{3 \times 31.8^2\,(31.8 - 24.8)}$$
$$= 1{,}988 \times .795 = 1{,}580 \text{ lbs.}$$

This result is attained by some labor which is practically unnecessary, because d should never be made less than *2/3 D*, and

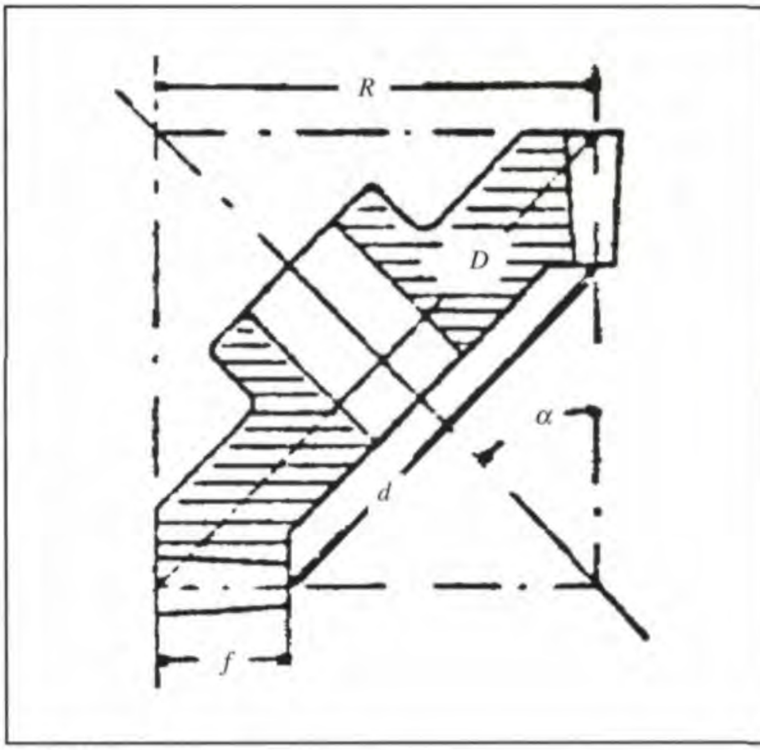

Plate II

when this rule is observed, the approximate formula,

$$W = s\,p\,f\,y\,\frac{d}{D} \qquad (6)$$

gives almost identical results. The reason for fixing a limit to the ratio $\frac{d}{D}$ is found in the fact that a further increase in face adds very slowly to the strength and increases very rapidly the difficulty of properly distributing the pressure transmitted. Where long-faced bevels are used, the teeth near the shaft are generally broken by improper fitting, or, when properly fitted, by the spring of the shaft or the yielding of its bearings, and, as the limit imposed gives about 70% of the strength attainable, by extending the face to the center, it is thought to be as liberal as experience can justify.

In presenting certain forms and proportions for teeth, on which Table 1 is founded, I am aware that other forms and proportions are in common use, which have some claims to recognition, but my chief object at present is to show that the strength of gearing can be reduced to a rational basis of comparison on which all authorities may ultimately unite. ■