Introduction

Analytical methods for determining the gear fillet profile (trochoid) have been well documented. Khiralla (Ref. 1) described methods for calculating the fillet profile of hobbed and shaped spur gears. Colbourne (Ref. 2) provided equations for calculating the trochoid of both involute and non-involute gears generated by rack or shaper tools. The MAAG Gear Handbook (Ref. 3) also provided equations for calculating trochoids generated with rack-type tools that have circular tool tips. Vijayakar, et al. (Ref. 4) presented a method of determining spur gear tooth profiles using an arbitrary rack. The above mentioned are only samples of many published works. However, the method for determining the trochoid of a helical gear generated with a shaper tool is not widely published. This article presents an intuitive algorithm where the fillet profile of a shaper-tool-generated external or internal helical gear can be calculated.

A shaper tool generating a gear can be visualized as a gear set meshing with zero backlash. The algorithm in this article is based on a shaper tool in tight mesh with a semi-finished helical gear. The semi-finished gear geometry was used for calculation because the shaper tool used as the semi-finishing tool is usually the one that generates the trochoid. However, if the shaper cutter is the finishing tool, the algorithm presented will also work by letting the finishing stock equal zero. The trochoid of a spur gear can also be calculated by letting the helix angle equal zero.

The shaper tool used in this algorithm may have a different reference normal pressure angle than that of the gear. A necessary condition for a shaper tool to generate the correct involute profile on a gear is that both the tool and the gear must have...
equal normal base pitches. This article stipulates that the axis of
the shaper tool and the gear are parallel, which is often true for
gear shaping. Consequently, the shaper tool and the gear must
also have equal base helix angles.

Although the algorithm is based on the shaper cutter as a
generating tool, the presented method can also be used to cal-
culate a trochoid generated with a hob or a rack-type tool if the
number of the shaper teeth is large (e.g. 10,000).

Symbols and Conventions
The symbols are defined where first used. This article tries
to adhere to the following rules in subscript usage:
• Symbols related to tool geometry have subscript “0”.
• No subscript is used for symbols related to the gear.
• Subscript “n” is used for measurements in the normal
plane.
• Subscript “r” is used for symbols related to the semi-fin-
ished gear.
• Subscript “g” is used for symbols related to the generat-

ing pitch circle.

When dual signs are used in an equation (e.g. ±), the upper
sign is for external gears and the lower one for internal gears.

Non-italicized uppercase symbols are used to designate
points on the shaper tool, the gear, or other points of interest.
Points are also represented as the coordinates (x, y). The length
of a vector (e.g. R) is represented as ||R||.

Coordinate System
The reference position of a shaper tool generating a gear is
depicted in Figure 1 for external gear shaping and Figure 2 for
internal.

The following coordinate system and sign conventions are
followed:
• A standard cartesian coordinate system is used. The cen-
ter of the shaper tool O₀ is (0,0).
• The reference position of the shaper tool is with one
of its teeth aligned with the y-axis. The end of the shaper
tooth points in the –y direction.
• The center of the gear, O₉, is also on the y-axis with one
of the tooth spaces aligned with the y-axis. The opening
of the tooth space is in the +y direction.
• Angular measures, related to tool or gear rotation or
location of a point, have signs. Counterclockwise rota-
tion from the reference line is positive, and clockwise is
negative.

Shaper Tool and Gear Geometry
The following are required tool and gear data for calculat-
ing the trochoid:

Shaper tool data:

\[ P_{nd0} \]  is the reference normal diametral pitch, tool (in.⁻¹)
\[ n₀ \]  is the number of teeth, tool
\[ \phi_{n₀} \]  is the reference normal pressure angle, tool
\[ \psi₀ \]  is the reference helix angle, tool
\[ s_{n₀} \]  is the reference normal circular thickness, tool (in.)
\[ d_{a₀} \]  is the outside diameter, tool (in.)
\[ \rho₀ \]  is the tool tip radius (in.)
\[ \delta₀ \]  is the protuberance (in.)

Gear data:

\[ P_{nd} \]  is the reference normal diametral pitch, gear (in.⁻¹)
\[ n \]  is the number of teeth, gear
\[ \phi_n \]  is the reference normal pressure angle, gear
\[ \psi \]  is the reference helix angle, gear
\[ s_n \]  is the reference normal circular thickness, gear (in.)
\[ \mu_s \]  is the stock allowance per flank, gear (in.), defined on the
reference pitch circle (not along the base tangent).

Basic Shaper Tool and Gear Geometry
The following equations calculate the basic tool and gear
Standard transverse pressure angle of tool, $\phi_0$

$$\psi_0 = \arctan \left( \frac{\tan \phi_0}{\cos \psi_0} \right)$$

Standard reference pitch radius of tool, $r_0$ (in.)

$$r_0 = \frac{n_0}{2 \pi \rho_{06}} \cos \psi_0$$

Base radius of tool, $r_{b0}$ (in.)

$$r_{b0} = r_0 \cos \phi_0$$

Reference transverse circular thickness of tool, $S_0$ (in.)

$$S_0 = \frac{s_{06}}{\cos \psi_0}$$

Transverse base pitch of tool, $P_{b0}$ (in.)

$$P_{b0} = \frac{2 \pi r_{b0}}{n_0}$$

Normal base pitch of tool, $P_{n0}$ (in.)

$$P_{n0} = \frac{n \cos \phi_0}{P_{n0}}$$

Base helix angle of tool, $\psi_{b0}$

$$\psi_{b0} = \arccos \left( \frac{P_{n0}}{P_{b0}} \right)$$

Base circular thickness of tool, $s_{b0}$ (in.)

$$s_{b0} = 2 r_{b0} \left( s_{06} + \inv \psi_{b0} \right)$$

where

$$\inv \alpha = \tan \alpha - \alpha$$

Standard reference pitch radius of gear, $r$ (in.)

$$r = \frac{n}{2 \pi \rho_{n0} \cos \psi}$$

Base radius of semi-finished gear, $r_{b}$ (in.)

$$r_{b} = \frac{n}{n_0} \rho_{b0}$$

The helix angle at standard pitch radius of semi-finished gear, $\psi_0$

$$\psi_{0} = \arctan \left( \frac{r \tan \psi_{b0}}{r_{b0}} \right)$$

Transverse pressure angle at reference pitch radius of semi-finished gear, $\phi_{0}$

$$\phi_{0} = \arccos \left( \frac{r_{b0}}{r_{b}} \right)$$

Transverse circular thickness of semi-finished gear, $s_{r}$ (in.)

$$s_{r} = \frac{s_{0} + 2 \mu_{r}}{\cos \psi_{r}}$$

Base circular thickness of semi-finished gear, $s_{b}$ (in.)

$$s_{b} = 2 r_{b} \left( \frac{1}{r_{b}} \pm \inv \phi_{r} \right)$$

**Center of Tool Tip on a Shaper Tool**

A shaper tool for gear semi-finishing usually has protuberance. It generates undercut on a gear, so that the finishing tool only needs to machine the involute profile of the gear. To obtain the designed amount of protuberance on a shaper tool, the tool tip is made tangent to the involute profile that is temporarily formed by increasing the shaper tooth thickness to include the protuberance (see Fig. 3). The tangent point, common to the tool tip and the involute profile, will be referred to as the profile tangent point, $P_0$. When the temporarily formed involute profile is removed, the shaper tool will have the designed amount of protuberance.

The shaper tool tip is also made tangent to the outside diameter of the tool (see Fig. 4) so that the transition from the outside diameter to the tool tip will be smooth. The common tangent point on the shaper tool tip and the outside diameter of the tool will be referred to as the end tangent point, $E_r$.

The following are the required data for calculating the center of the shaper tool tip:

- $d_{w0}$ is the outside diameter, tool (in.)
- $s_{w0}$ is the base circular thickness, tool (in.)
- $\rho_0$ is the tool tip radius (in.)
- $\delta_0$ is the protuberance (in.)
- $\psi_{b0}$ is the base helix angle, tool
- $\psi_0$ is the reference helix angle, tool

The base circular thickness of the involute profile, formed.
by increasing the shaper tool tooth thickness to include the protuberance, $\delta_{t0,pr}$

$$x_{0,pr} = x_{0} + \frac{2\delta_{0}}{\cos \psi_{0}} \tag{15}$$

Coordinates of the center of the tool tip, $S_{0}$

$$S_{0} = (r_{0}, \sin \lambda_{0}, -r_{0}, \cos \lambda_{0}) \tag{16}$$

where

$r_{0}$ is the tool radius to the center of the tool tip (in.)

$\lambda_{0}$ is the offset angle of the tool tip.

For a shaper tool with full tip radius, $\lambda_{0}$ will equal zero.

Coordinates of the profile tangent point, $P_{0}$, are

$$P_{0} = S_{0} + (\rho_{0}, \cos \theta_{p0}, \rho_{0}, \sin \theta_{p0}) \tag{17}$$

where

$\theta_{p0}$ is the auxiliary angle that locates $P_{0}$. The angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. $\theta_{p0}$ will usually have a negative value.

The tool radius to profile tangent point, $r_{p0}$ (in.), is

$$r_{p0} = || P_{0} || \tag{18}$$

The transverse pressure angle, $\phi_{p0'}$ at $P_{0}$ is

$$\phi_{p0} = \arccos\left(\frac{r_{0}}{r_{p0}}\right) \tag{19}$$

The tangent angle, $\alpha_{p0'}$, at $P_{0}$ (the derivation of Equation 20 is given in Appendix A) is

$$\alpha_{p0} = \arctan\left(-\frac{\cos \psi_{0}}{\tan \theta_{p0}}\right) \tag{20}$$

The angle between the y-axis and the radius to the profile tangent point, $\zeta_{p0'}$, is

$$\zeta_{p0} = \frac{x_{0,pr}}{2r_{0}} \tag{21}$$

The coordinates of the end tangent point, $E_{0}$, are

$$E_{0} = S_{0} + (\rho_{0}, \cos \theta_{e0}, \rho_{0}, \sin \theta_{e0}) \tag{22}$$

where

$\theta_{e0}$ is the auxiliary angle that locates $E_{0}$. The angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. $\theta_{e0}$ will usually have a negative value.

The angle of tangent, $\alpha_{e0'}$, at the end tangent point, $E_{0}$, is

$$\alpha_{e0} = \arctan\left(-\frac{\cos \psi_{0}}{\tan \theta_{e0}}\right) \tag{23}$$

The following are conditions for the tool tip to position properly on a shaper tool tooth:

1) The profile tangent point, $P_{0'}$, on the tool tip must also be a point on the involute profile that includes the protuberance, thus

$$\alpha_{p0} + \phi_{p0} - \zeta_{p0} - \frac{\pi}{2} = 0 \tag{25}$$

2) The angle, $\zeta_{p0'}$, subtended by one half of the transverse circular thickness of the involute curve (include the tool protuberance) at $P_{0'}$ must equal the angle formed by the y-axis and the line connecting the center of the tool to $P_{0'}$

$$\zeta_{p0} - \arcsin\left(\frac{x_{p0}}{r_{p0}}\right) = 0 \tag{26}$$

3) The end tangent point must also be a point on the outside diameter of the shaper tool, thus

$$r_{e0} - \frac{d_{u}}{2} = 0 \tag{27}$$

4) The tangent angle, $\alpha_{e0'}$, at the end tangent point, $E_{0}$, must equal the angle formed by the y-axis and the line connecting the center of the tool to $E_{0}$

$$\alpha_{e0} = \arcsin\left(\frac{x_{e0}}{r_{e0}}\right) = 0 \tag{28}$$

Figure 4—End of tool tip (with helix angle exaggerated).
Equations 25–28 must all be satisfied for the tool tip to be correctly positioned on a shaper tool tooth. The variables to be determined are \( t_{S_{00}}, \lambda_{S_{00}}, \theta_{P_{00}} \) and \( \theta_{E_{00}} \). Since the systems of the equations are transcendental and cannot be solved directly, the Newton’s method is used to calculate the roots for Equations 25–28.

**Solving the System of Non-linear Equations for Center of Tool Tip**

For simplicity, rewrite Equations 25–28 as generic vector equations in the form

\[
F(X) = 0
\]  

where

\[
F(X) = (f_1(X), f_2(X), f_3(X), f_4(X))^T
\]

\[
0 = (0,0,0,0)^T
\]

\[
X = (x_1, x_2, x_3, x_4)^T
\]

\[
J = (r_{S_{00}}, \lambda_{S_{00}}, \theta_{P_{00}}, \theta_{E_{00}})^T
\]

The Newton’s iteration equation (Ref. 6) is written as

\[
X_1 = X + \delta X
\]  

where \( \delta X \) satisfies the following system of linear equations

\[
J \cdot \delta X = -F(X)
\]  

where

- \( X_1 \) is the vector of the new roots for the next iteration
- \( X \) is the vector of current roots
- \( \delta X \) is the vector of Newton’s steps for the next iteration
- \( J \) is the Jacobian matrix

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_4} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \ldots & \frac{\partial f_4}{\partial x_4}
\end{bmatrix}
\]

\( \frac{\partial f_i}{\partial x_j} \) is the partial derivative of the \( i \)-th equation with respect to the \( j \)-th variable.

The Newton’s iteration procedure is described below:

1. Select a set of initial guess values for the new root, \( X_1 \). The following are the suggested values:

\[
\begin{align*}
x_i(t_{S_{00}}) &= \frac{d_{a0}}{2} - \rho_0 \\
x_j(t_{S_{00}}) &= 0.0175 \\
x_j(\theta_{P_{00}}) &= -\theta_{a0} \\
x_j(\theta_{E_{00}}) &= -1.4835
\end{align*}
\]

2. Select the initial Newton’s steps, \( \delta X \). The following values work satisfactorily:

\[
\delta X = (0.01, 0.01, 0.01, 0.01)^T
\]

3. Evaluate the system of non-linear equations (see Eq. 30) at the new root, \( F(X_1) \).

4. Calculate the error \( \text{ERR}(X_1) \) (see Eq. 37).

5. The iteration is terminated, if \( \text{ERR}(X_1) \leq 10^{-10} \) or if a predetermined number of iterations has been reached. Otherwise, continue with the next step.

6. Save the new roots as the current roots, so that a new set of roots can be calculated

\[
X = X_1
\]

7. Calculate the Jacobian matrix, column by column, starting with column one using Equation 36. Repeat the calculation procedure for the remaining columns until the Jacobian matrix is completed (see Eq. 35).

8. Solve the system of linear equations (see Eq. 34) for the next set of the Newton’s steps, \( \delta X \).

9. Calculate new roots, \( X_1 \), using Equation 33.

10. Repeat steps 3–9 until step 5 is satisfied.

The system of linear equations in step 8 (see Eq. 34) can be solved by inverting the Jacobian matrix or by using one of many

\[
\frac{\partial f_i}{\partial x_j} = \frac{f_i(X + \Delta X_j) - f_i(X)}{\Delta X_j}
\]  

where

- \( i \) is the \( i \)-th row of the Jacobian matrix
- \( j \) is the \( j \)-th column of the Jacobian matrix
- \( \Delta X_j \) is a vector with its \( j \)-th element equal to the \( j \)-th element of the current Newton’s step, \( \delta X \), and all remaining elements equal 0.

For each iteration, the sum of the absolute values of the functions (errors) is calculated.

\[
\text{ERR}(X_1) = \sum_i |f_i(X_1)|
\]  

The Newton’s iteration procedure is terminated when the error (see Eq. 37) becomes smaller than a predetermined tolerance, or when a predetermined number of iterations has been reached.
Generating Pressure Angle and Center Distance

The generating pressure angle and the center distance are based on tight meshing of a shaper tool with a semi-finished gear. The involute function of the generating pressure angle, \( \tan \phi_g \), is given by the following equation (the derivation of Equation 39 is given in Appendix A):

\[
\tan \phi_g = \frac{s_{br} + s_{br} P_{br}}{2(r_{br} \pm r_{br})}
\]

(39)

where

\( s_{br} \) is the base circular thickness of the tool (in.)
\( s_{br} \) is the base circular thickness of the semi-finished gear (in.)
\( P_{br} \) is the transverse base pitch of the tool (in.)
\( r_{br} \) is the base radius of the tool (in.)
\( r_{br} \) is the base radius of the semi-finished gear (in.)

The generating pressure angle, \( \phi_g \), can be calculated by taking the arc of the involute function (Ref. 5). The generating center distance, \( c_g \), is given by:

\[
c_g = \frac{r_{br} \pm r_{br}}{\cos \phi_g}
\]

(40)

The generating pitch radius of the shaper tool, \( r_{\psi} \), is

\[
r_{\psi} = \frac{r_{br}}{\cos \phi_g}
\]

(41)

The generating pitch radius of the gear, \( r_{\psi} \), is

\[
r_{\psi} = \frac{r_{br} n}{\sin \theta_{\psi}}
\]

(42)

Determination of Shaper-Tool-Generated Fillet Profile

Conjugate point of an arbitrary point on a shaper tool tip.

The fillet profile (trochoid) of a helical gear is generated by the tool tip of a shaper tool. This section describes the procedure for calculating a point on the trochoid that is conjugate to an arbitrary point on the shaper tool tip, \( X_0 \) (see Fig. 5).

The coordinates of an arbitrary point, \( X_{\psi} \), on the tool tip are

\[
X_0 = S_0 + (\rho_0 \cos \theta_{\psi}, \rho_0 \sin \theta_{\psi})
\]

(43)

where

\( S_0 \) are the coordinates of the center of tool tip (in., in.)
\( \rho_0 \) is the tool edge radius (in.)
\( \theta_{\psi} \) is the auxiliary angle that locates an arbitrary point on the tool tip. This angle is measured in the normal plane, clockwise from the horizontal axis of the tool tip. \( \theta_{\psi} \) will usually have a negative value.

The slope, \( m_{\psi} \), of the normal passing through \( X_0 \) is

\[
m_{\psi} = \frac{\tan \theta_{\psi}}{\cos \psi_0}
\]

(44)

The derivation of Equation 44 is given in Appendix A.

Note: For a shaper tool with non-elliptical tool tip, Equations 43 and 44 should be bypassed and the actual tool tip geometry, \( X_0 \) and \( m_{\psi} \), should be used for the subsequent calculations.

The normal at \( X_0 \) can be expressed as a linear equation:

\[
y = m_{\psi}(x - x_0) + y_0
\]

(45)

where

\( x_0 \) is the x-coordinate of \( X_0 \)
\( y_0 \) is the y-coordinate of \( X_0 \)

When extended, the normal will intersect the generating pitch circle of the shaper tool at point \( G_0 \) (see Fig. 5). The x-coordinate of the intersection point can be calculated as:

\[
x_{G_0} = m_{\psi} k_2 + \sqrt{r_{\psi}^2 k_1 - k_2^2}
\]

(46)

where

\( r_{\psi} \) is the generating pitch radius, tool (in.)
\( k_1 \) is a temporary variable
\( k_2 \) is a temporary variable (in.)

\[
k_1 = m_{\psi}^2 + 1
\]

(47)

\[
k_2 = m_{\psi} x_0 - y_0
\]

(48)

The angle, \( \phi_{\psi} \), formed between the y-axis and the tool radius at the intersection point, \( G_0 \), is
Figure 6—The arbitrary point \( X_0 \) and the normal after rotating the shaper tool for an angle \( -\xi_0 \).

\[ \xi_0 = \arcsin \left( \frac{x_{G0}}{r_{G0}} \right) \quad (49) \]

Note: The angle \( \xi_0 \) may be positive or negative. If \( G_0 \) is on the left side of the y-axis, \( x_{G0} \) (see Eq. 49) will be negative, and so will \( \xi_0 \). On the other hand, if \( G_0 \) is on the right side of the y-axis, \( \xi_0 \) will have a positive value.

To find the conjugate point of \( X_0 \), the shaper tool is rotated from its reference position (see Fig. 6) by an angle, \( -\xi_0 \). The arbitrary point \( X_0 \) will rotate to a new position, \( X_0' \).

\[ X_0' = M(-\xi_0)X_0 \quad (50) \]

where

\[ M(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \quad (51) \]

\( M(\varphi) \) is a rotation matrix. When multiplied to a vector, the vector would be rotated an angle \( \varphi \) about the origin (0,0). If \( \varphi > 0 \), the rotation is counterclockwise. Otherwise, the rotation is clockwise.
After rotating the cutter (see Fig. 6), the normal at the arbitrary tool tip point (now \(X_0')\) will pass through the generating pitch point \(G\), thus satisfying the law of conjugate action (Ref. 1):

\[ \zeta = \pm \xi_0 \frac{n_0}{r} \]  

(52)

To return the gear to its reference position, it is rotated an angle \(-\xi\) about the gear center \(O_G\). After rotating the gear, the common point \(X_0'\) will move to \(T_{x_0}\) (see Fig. 7), which is a point on the trochoid. \(T_{x_0}\) can be calculated as:

\[ T_{x_0} = M(-\xi)(X_0' - O_G) \]  

(53)

Note: In Equation 53, the origin of \(T_{x_0}\) (see Eq. 53) is the center of the gear \(O_G\), not the center of the tool.

Determination of a Shaper-Tool-Generated Fillet Profile.

The shaper tool discussed in this article can be divided into three zones (see Fig. 8):

1) Zone 1 is the portion of cutter profile that coincides with the outside diameter of the shaper tool. It starts from the outside diameter of the cutter on the y-axis, and ends at the end tangent point, \(E_0\). The tool profile in this zone generates the root circle of the gear. If the shaper tool has a full tip radius, Zone 1 reduces to a single point on the outside diameter of the tool.
2) Zone 2 is the elliptical tool tip starting at \(E_0\) and ends where the tool tip joins the main shaper tool profile (Zone 3).
3) Zone 3 is the main cutter profile that generates the involute profile on the semi-finished gear.

The shaper-tool-generated trochoid can be determined by calculating the conjugate points of the tool tip in Zone 2. Begin the calculation at \(E_0\) (see Fig. 8), and continue in small increments towards \(P_0\). The conjugate point of \(P_0\) will usually penetrate deepest from the surface of the involute profile (see Fig. 9). Continue the calculation procedure until the trochoid intersects the involute tool profile. Additional trochoid points can be calculated if desired.

Using Shaper Tool Algorithm to Calculate Fillet Profile of a Hobbed Gear

The tooth profile of a shaper tool with an infinite number of teeth will approach a rack. Naturally, if a shaper tool algorithm could handle an infinite number of tool teeth, a hobbed trochoid could be accurately approximated. Unfortunately, the shaper tool algorithm presented in this article does not allow for an infinite number of tool teeth. A shaper tool with a finite, but large number of teeth is permitted.

To investigate the feasibility of approximating a hobbed trochoid with the shaper algorithm using a shaper tool with a large number of teeth, a numerical example was calculated using Example 3.1.5 of AGMA 918-A93 (Ref. 8). The number

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Table 1—Comparison of a Hobbed Pinion Fillet Profile (Ex. 3.1.5 - AGMA 918-A93) with Fillet Profiles Generated with 100-, 1,000-, and 10,000-Tooth Shaper Tools.

<table>
<thead>
<tr>
<th>Description</th>
<th>Gear data</th>
<th>Tool data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hobbed</td>
<td>100T-Shaper</td>
</tr>
<tr>
<td>Normal diametral pitch</td>
<td>in.(^{-1})</td>
<td>12</td>
</tr>
<tr>
<td>Number of teeth</td>
<td>35</td>
<td>NA</td>
</tr>
<tr>
<td>Reference normal pressure angle</td>
<td>deg.</td>
<td>20</td>
</tr>
<tr>
<td>Reference helix angle</td>
<td>deg.</td>
<td>22.109</td>
</tr>
<tr>
<td>Outside diameter (or hob addendum)</td>
<td>in.</td>
<td>3.3686</td>
</tr>
<tr>
<td>Reference normal circular thickness</td>
<td>in.</td>
<td>0.1501</td>
</tr>
<tr>
<td>Stock allowance</td>
<td>in.</td>
<td>0.001</td>
</tr>
<tr>
<td>Tool tip radius</td>
<td>in.</td>
<td>0.0100</td>
</tr>
<tr>
<td>Protuberance</td>
<td>in.</td>
<td>0.0025</td>
</tr>
</tbody>
</table>

Comparison of the calculated fillet profile

Maximum difference between hobbed & shaped profiles

|                        | in.       | NA           | 0.001901 | 0.000111 | 0.0000111 |

Comparison of form diameter (SOI)

|                       | in.       | NA           | 3.040483 | 3.050692 | 3.041641  |
|                       |          |              | 3.040600 |

Difference between hobbed & shaper-generated form diameters

|                       | in.       | NA           | 0.010209 | 0.001158 | 0.000117  |
of shaper tool teeth used were 100, 1,000 and 10,000. Table 1 compares the distances between the trochoid curves generated with the shaper cutters and the one generated with a hob. The form diameters or the start of involute (SOI) (to be discussed in the section “Calculating Form Diameter,” below) based on the shaper tool were also compared to that generated with a hob. The calculated SOI’s using the shaper tool algorithm compared well with those using other software.

Checking Gear Finishing Stock

For gears finished by grinding or shaving, the semi-finishing tool is usually designed with protuberance that would generate an undercut in the gear. The protuberance provides stock for finishing operations. The form diameter (SOI) of the finished gear must be smaller than the start of active profile (SAP) of the gear when the gear meshes with the mate. The algorithm presented in this article can verify if a semi-finishing tool would provide sufficient finishing stock on the gear while keeping the SOI smaller than the SAP.

Consider a helical gear set with the basic geometry given in Table 3. The initial pinion hob (A) design used the same standard reference pressure angle, 20°, as the part. Consequently, the calculated SOI (4.4873”) was larger than the SAP (4.4788”).
Table 2—Comparison of the Form Diameters Calculated Using the Proposed Algorithm and Other Software.

<table>
<thead>
<tr>
<th>Description</th>
<th>Example 3-1-1</th>
<th>Example 3-1-3</th>
<th>Example 3-1-9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gear data</td>
<td>Pinion Gear</td>
<td>Pinion Gear</td>
<td>Pinion Gear</td>
</tr>
<tr>
<td>Gear type</td>
<td>Spur</td>
<td>Single helical</td>
<td>Internal helical</td>
</tr>
<tr>
<td>Normal diametral pitch</td>
<td>in. (^{-1})</td>
<td>5 5</td>
<td>6 6</td>
</tr>
<tr>
<td>Number of teeth</td>
<td>51 104</td>
<td>21 86</td>
<td>24 69</td>
</tr>
<tr>
<td>Ref. norm. press. angle</td>
<td>deg.</td>
<td>20.0000 20.0000</td>
<td>20.0000 20.0000</td>
</tr>
<tr>
<td>Standard helix angle</td>
<td>deg.</td>
<td>0.0000 0.0000</td>
<td>15.0000 15.0000</td>
</tr>
<tr>
<td>Normal circular thickness</td>
<td>in.</td>
<td>0.326267 0.293451</td>
<td>0.322622 0.257794</td>
</tr>
<tr>
<td>Stock allowance</td>
<td>in.</td>
<td>0.008000 0.008000</td>
<td>0.005300 0.005300</td>
</tr>
</tbody>
</table>

Table 3—Comparison of Form Diameters of a Pinion Generated with Normal Lead and Short Lead Hobs.

<table>
<thead>
<tr>
<th>Description</th>
<th>Unit</th>
<th>Oper. Cntr. Dist. 14.500 in.</th>
<th>Pinion Hob A (normal lead)</th>
<th>Pinion Hob B (short lead)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal diametral pitch</td>
<td>in. (^{-1})</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.1211</td>
</tr>
<tr>
<td>Number of teeth</td>
<td>93</td>
<td>18</td>
<td>10,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Ref. norm. pressure angle (part or hob)</td>
<td>deg.</td>
<td>20</td>
<td>20</td>
<td>14.5</td>
</tr>
<tr>
<td>Reference helix angle</td>
<td>deg.</td>
<td>15.1560</td>
<td>15.1560</td>
<td>14.7003</td>
</tr>
<tr>
<td>Outside diameter (or hob addendum)</td>
<td>in.</td>
<td>24.5840 5.4160</td>
<td>0.3372 0.1373</td>
<td></td>
</tr>
<tr>
<td>Reference normal circular thickness</td>
<td>in.</td>
<td>0.3874 0.4812</td>
<td>0.3889 0.2419</td>
<td></td>
</tr>
<tr>
<td>Stock allowance per flank</td>
<td>in.</td>
<td>0.0050 0.0050</td>
<td>NA NA</td>
<td></td>
</tr>
<tr>
<td>Tool tip radius</td>
<td>in.</td>
<td>NA</td>
<td>0.0900 0.0900</td>
<td></td>
</tr>
<tr>
<td>Protruberance</td>
<td>in.</td>
<td>0.0070</td>
<td>0.0070</td>
<td></td>
</tr>
</tbody>
</table>

Therefore, hob (A) does not provide the required grinding stock while keeping the SOI below the required SAP. In order to push the SOI closer to the root diameter, a short lead hob (B) was designed. The hob had a 14.5° reference normal pressure angle. The calculated SOI based on the short lead hob (B) was 4.4550", smaller than the SAP. Hob B provided the required finishing stock with satisfactory SOI (see Fig. 13).

**Conclusions**

A method for determining the shaper-tool-generated fillet profile (trochoid) was presented. The method is applicable to both external and internal helical gears. The algorithm is based on a class of shaper tool that has an involute main profile and elliptical tool tip in the transverse plane. However, the algorithm will also work for a shaper tool with other tool tip geometries, provided the coordinates and the normal of the tool tip profile are known.

The shaper tool algorithm can also approximate the trochoid generated with a rack-type tool if the number of shaper tool teeth is large. The numerical examples showed that a trochoid curve generated with a 10,000-tooth shaper tool can approximate that generated with a hob with small error.

The algorithm presented in this article does not require the
tool and the gear to have equal reference normal pressure angle. Consequently, a trochoid generated with a non-standard cutter such as a short lead hob can also be calculated.

Examples for the form diameter (SOI) calculation and the finishing stock analysis were provided using the shaper tool algorithm presented.

A computer program was developed using the algorithm described in this article. The calculated form diameters (SOI’s) for both external and internal gears compare well to those calculated with other gear software. An internal spur gear was used to verify the shaper tool algorithm.

Acknowledgements

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Kevin Acheson, The Gear Works, helped with the shaper tool geometry and supplied an internal spur gear sample for checking the output of the computer program. He also created a hobbed example that resulted in rewriting a part of the shaper tool algorithm. Appreciations are also due to Robert F. Wasilewski, Arrow Gear Co., for calculating some of the numerical examples in this article using the other software.

Finally, I would like to thank others, too many to mention individually, that have helped with this project or offered valuable comments that were, in one way or another, incorporated into this article.

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References

Appendix A—Derivation of Equations

The tangent and the normal of an arbitrary point on the shaper tool tip (Equations 20 and 44). The shaper tool tip considered in this article is circular in the normal plane and elliptical in the transverse plane, as shown in Figure A1. (Ref. A1). The coordinates of an arbitrary tool tip point X₀ (related to the center of the tool tip) in the transverse plane can be calculated as

\[ x_{X₀} = \rho₀ \cos\psi_0 \tan\theta_{X₀₀} \]  \hspace{1cm} (A.1)

\[ y_{X₀} = \rho₀ \sin\theta_{X₀₀} \]  \hspace{1cm} (A.2)

where

\[ \rho₀ \]  is the tool tip radius and

\[ \theta_{X₀₀} \]  is the auxiliary angle for point X₀ measured clockwise from the horizontal axis.

Differentiating Equation A.1 and Equation A.2 with respect to the auxiliary angle \( \theta_{X₀₀} \) we get

\[ dx_{X₀} = -\rho₀ \sin\psi_0 \cos\theta_{X₀₀} \frac{d\theta_{X₀₀}}{dx_{X₀}} \]  \hspace{1cm} (A.3)

\[ dy_{X₀} = \rho₀ \cos\theta_{X₀₀} \cos\psi_0 \frac{d\theta_{X₀₀}}{dx_{X₀}} \]  \hspace{1cm} (A.4)

The slope of the tangent at point X₀ can be calculated as

\[ \tan\alpha_{X₀} = \frac{dy_{X₀}}{dx_{X₀}} = -\frac{\cos\psi_0}{\tan\theta_{X₀₀}} \]  \hspace{1cm} (A.5)
Similarly, the slope of the profile tangent point, \( P_0 \) (see the section “Center of tool tip on a shaper tool,” above), can be calculated as

$$\tan \alpha_{P_0} = -\frac{\cos \psi_0}{\tan \theta_{P_0}}$$  \hspace{1cm} (A.6)$$

Taking an arc tangent on both sides of Equation A.6 completes the derivation for Equation 20.

$$\alpha_{P_0} = \arctan \left( -\frac{\cos \psi_0}{\tan \theta_{P_0}} \right)$$  \hspace{1cm} (A.7)$$

The normal at the given arbitrary point on a shaper tool tip is perpendicular to the tangent. Therefore, the slope of the normal, \( m_{X_0} \) (see Eq. 44), is:

$$m_{X_0} = \frac{-1.0}{\tan \alpha_{X_0}} = \frac{\tan \theta_{X_0}}{\cos \psi_0}$$  \hspace{1cm} (A.8)$$

Generating pressure angle (Equation 39). The generating pressure angle is based on tight meshing of a shaper tool with a semi-finished gear. The derivation of the generating pressure angle equation is similar to the one given in 86 FTM 1 (Ref. A2).

The following tool and gear data are given:

- \( s_{b0} \) is the transverse base circular thickness, tool (in.);
- \( r_{b0} \) is the base radius, tool (in.);
- \( s_{b1} \) is the transverse base circular thickness, semi-finished gear (in.). If shaping is the finishing operation, the base circular thickness for the finished gear should be used; and
- \( r_{b1} \) is the base radius, semi-finished gear (in.).

The sum of the transverse circular thickness of the tool and the gear equals the circular pitch at the generating pitch circle.

$$p_{g0} = s_{g0} + s_{g1}$$  \hspace{1cm} (A.9)$$

where

- \( p_{g0} \) is the transverse circular pitch at the generating pitch circle;
- \( s_{g0} \) is the transverse circular thickness at the generating pitch circle, tool (in.); and
- \( s_{g1} \) is the transverse circular thickness at the generating pitch circle, gear (semi-finished) (in.).

The circular thicknesses of tool and semi-finished gear at the generating pitch circle can be calculated as

$$s_{g0} = 2r_{g0} \left( \frac{s_{b0}}{2r_{b0}} - \text{inv} \phi_{g0} \right)$$  \hspace{1cm} (A.10)$$

$$s_{g1} = 2r_{g1} \left( \frac{s_{b1}}{2r_{b1}} - \text{inv} \phi_{g1} \right)$$  \hspace{1cm} (A.11)$$

where

- \( r_{g0} \) is the generating pitch radius of the shaper tool (in.);
- \( r_{g1} \) is the generating pitch radius of the semi-finished gear.

Substituting Equation A.9 and Equation A.10 into Equation A.8 and dividing both sides of the new equation by \( 2r_{g0} \) we get

$$\frac{p_{g0}}{2r_{g0}} = \frac{s_{g0}}{2r_{b0}} - \text{inv} \phi_{g0} + \frac{r_{b0} s_{b0}}{2r_{b0}} \frac{s_{g1}}{r_{g1}} - \frac{r_{g1}}{r_{b0}} \text{inv} \phi_{g1}$$  \hspace{1cm} (A.12)$$

using the following established relationships

$$\frac{p_{g0}}{2r_{g0}} = \frac{p_{b0}}{2r_{b0}}$$  \hspace{1cm} (A.13)$$

$$\frac{r_{g1}}{r_{g0}} = \frac{r_{b1}}{r_{b0}}$$  \hspace{1cm} (A.14)$$

where

- \( p_{b0} \) is the transverse base circular pitch, tool (in.).

Substituting Equation A.12 and Equation A.13 into Equation A.11, we get

$$\frac{p_{b0}}{2r_{b0}} = \frac{s_{b0}}{2r_{b0}} - \text{inv} \phi_{g0} + \frac{r_{b0} s_{b0}}{2r_{b0}} \frac{s_{b1}}{r_{b1}} - \frac{r_{b1}}{r_{b0}} \text{inv} \phi_{g1}$$  \hspace{1cm} (A.15)$$

Multiply both sides of Equation A.14, by \( 2r_{b0} \) and solve for \( \text{inv} \phi_{g} \) (Eq. 39)

$$\text{inv} \phi_{g} = \frac{s_{b0} + s_{b1} - p_{b0}}{2(r_{b0} + r_{b1})}$$  \hspace{1cm} (A.16)$$

Bibliography
