

# Gear Transmission Density Maximization

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## Management Summary

Maximization of gear transmission density is important in that it delivers increased output torque within given dimensional constraints. This is critical, for instance, in racing gearboxes or in reducing size and weight of aerospace gear drives. It can also yield reduced costs for automotive and consumer product gear trains, for example. There are several ways to increase gear drive load capacity, including advanced design, materials and technologies.

This paper presents an approach that provides optimization of both gearbox kinematic arrangement and gear tooth geometry to achieve a high-density gear transmission. It introduces dimensionless gearbox volume functions that can be minimized by the internal gear ratio optimization. Different gearbox arrangements are analyzed to define a minimum of the volume functions. Application of asymmetric gear tooth profiles for power density maximization is also considered.

### Introduction: Volume Functions

The gearbox weight minimization software (Ref. 1) defines internal gear distribution for different gear drive arrangements. However, it does not take into account the gear volume utilization, i.e., the ratio of the actual gear volume to the pitch cylinder volume. It also is not applicable to multi-branch epicyclic gear drive arrangements (Fig. 6). Load capacity or transmission density is defined by a gear tooth's working flank surface durability, which is limited by allowable contact stress level. For a pair of mating gears, this can be described by the gear transmission density coefficient  $K_o$  (Ref. 2) that is equal to:

$$K_o = \frac{2 \times T_1}{d_{w1}^2 \times b_w} \times \frac{u \pm 1}{u} \quad (1)$$

where:

$T_1$	Driving pinion torque
$d_{w1}$	Pinion operating pitch diameter
$b_w$	Effective gear face width in mesh

$u = n_2/n_1$	Gear pair ratio
$n_1$	Driving pinion—number of teeth
$n_2$	Driven gear—number of teeth
“+”	External gear mesh
“–”	Internal gear mesh

Depending on the application, the gear pair transmission density coefficient  $K_o$  statistically varies about 0.5–4.0 MPa for commercial drives and about 4.0–12.0 MPa for aerospace, racing and automotive drives. The gear pair volume definition is illustrated in the Figure 1.

Weight of the pinion can be presented as:

$$w_1 = \rho \times V_1 \times K_{v1} \quad (2)$$

where:

$\rho$	Material density
$K_{v1}$	Volume utilization coefficient for the pinion
$V_1$	Operating pitch cylinder volume that is equal to:

$$V_1 = \frac{\pi}{4} \times d_{w1}^2 \times b_w \quad (3)$$

which, considering Equation 1, also can be presented as:

$$V_1 = \frac{\pi}{2} \times \frac{T_1}{K_o} \times \frac{u \pm 1}{u} \quad (4)$$

The operating pitch cylinder volume of the mating gear is

$$V_2 = \frac{\pi}{4} \times d_{w2}^2 \times b_w = u^2 \times V_1 \quad (5)$$

where:

$d_{w2}$  Gear operating pitch diameter

Total weight of two mating gears is:

$$w = w_1 + w_2 = \rho \times (V_1 \times K_{v1} + V_2 \times K_{v2}) \quad (6)$$

where:

$K_{v1}$  Volume utilization coefficient of the pinion  
 $K_{v2}$  Volume utilization coefficient of the mating gear

Then applying Equation 5:

$$w = \rho \times V_1 \times (K_{v1} + u^2 \times K_{v2}) \quad (7)$$

or applying Equation 4:

$$w = \rho \times \frac{\pi}{2} \times \frac{T_1}{K_o} \times F_v \quad (8)$$

where:

$F_v$  Dimensionless volume function

For the cylindrical pair of gears the volume function is:

$$F_v = F_{v1} + F_{v2} = \frac{u \pm 1}{u} \times (K_{v1} + u^2 \times K_{v2}) \quad (9)$$

where:

$$F_{v1} = \frac{u \pm 1}{u} \times K_{v1} \quad (10)$$

is the pinion volume function;

$$F_{v2} = (u \pm 1) \times u \times K_{v2} \quad (11)$$

is the mating gear volume function.

The epicyclic gear stage volume definition is illustrated in Figure 2. In this case the subscript indexes 1–3 ( $1, 2, 3$ ) are related to the sun gear, planet gear and ring gear accordingly.

The operating pitch cylinder volume of the

sun gear is defined by Equation 4 with a “+” sign because the sun gear is in the external mesh with the planet gear. The planet gear operating pitch cylinder volume is defined by Equation 5. The operating pitch cylinder volume of the ring gear is:

$$V_3 = \frac{\pi}{4} \times d_{w3}^2 \times b_{wi} = p^2 \times V_1 \times K_{bw} \quad (12)$$

where:

$d_{w3}$  Ring gear operating pitch diameter

$K_{bw} = b_{wi} / b_{we}$  Effective gear face width ratio in the epicyclic gear stage

$b_{we}$  Effective gear face width in the sun/planet gear mesh

$b_{wi}$  Effective gear face width in the planet/ring gear mesh

$p = |n_3 / n_1|$  Ring/sun gear ratio in the epicyclic stage

$n_3$  Ring gear number of teeth

Assuming the same density material for all gears, the total weight of gears in the epicyclic gear stage is:

$$w = w_1 + n_p \times w_2 + w_3 = \rho \times (V_1 \times K_{v1} + n_p \times V_2 \times K_{v2} + V_3 \times K_{v3}) \quad (13)$$

where:

$K_{v3}$  Volume utilization coefficient of the sun gear;

$n_p$  Number of planet gears.

Applying Equations 5 and 12, the total weight is:

$$w = \rho \times V_1 \times (K_{v1} + u^2 \times n_p \times K_{v2} + p^2 \times K_{v3} \times K_{bw}) \quad (14)$$

Then considering Equation 4, the epicyclic gear stage volume function is:

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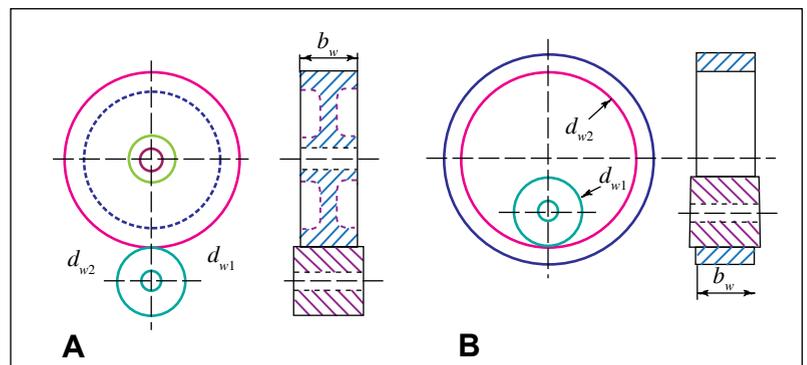


Figure 1—Gear pair volume definition: A = external gearing; B = internal gearing.

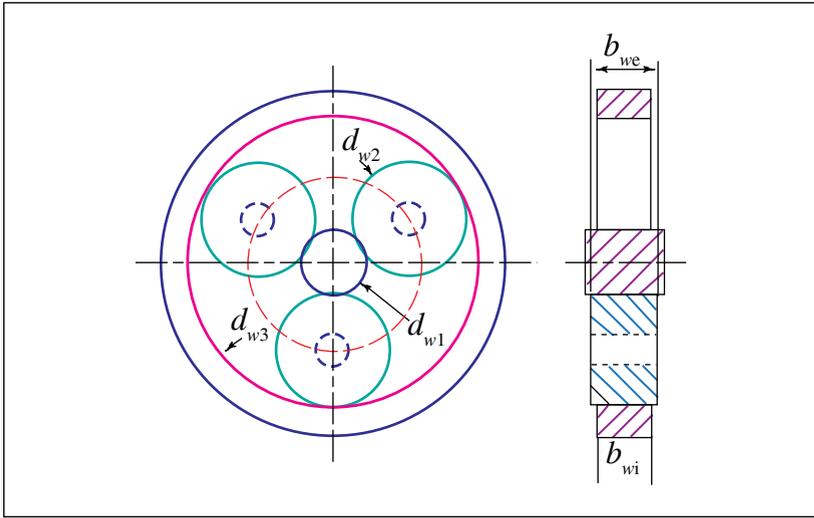


Figure 2—Epicyclic gear stage volume definition.

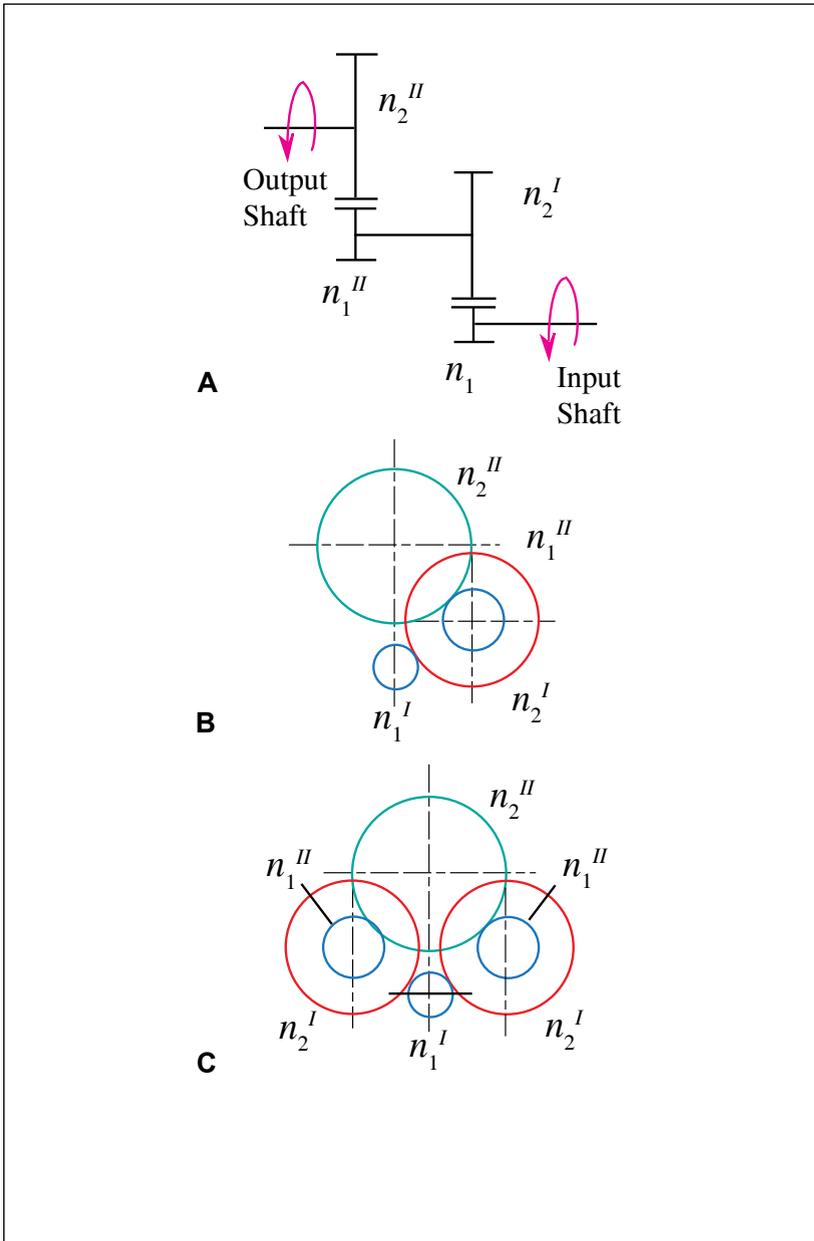


Figure 3—A = two-stage external gear train; B = one-branch transmission; C = two-branch transmission.

$$F_{ve} = F_{ve1} + n_p \times F_{ve2} + F_{ve3} = \frac{u+1}{u \times n_p} \times (K_{v1} + u^2 \times n_p \times K_{v2} + p^2 \times K_{v3} \times K_{bw}) \quad (15)$$

where:

$$F_{ve1} = \frac{u+1}{u \times n_p} \times K_{v1} \quad (16)$$

is the sun gear volume function;

$$F_{ve2} = \frac{u+1}{n_p} \times u \times K_{v2} \quad (17)$$

is the planet gear volume function,

$$F_{ve3} = \frac{u+1}{u \times n_p} \times p^2 \times K_{v3} \times K_{bw} \quad (18)$$

is the planet gear volume function.

The more planet gears in the epicyclic gear stage, the lower its volume function and more compact the gearbox. However, the selected number of planet gears must exclude the interference between them.

The volume utilization coefficients  $K_v$  depend on the gear body shape (solid body or with central and lightening holes, rim, web, spokes, etc.) and, for driving pinions (sun gears), varies approximately in a range of 0.8–1.0; for driven (or planet) gears, 0.3–0.7; and for internal (or ring) gears, 0.05–0.1.

Unlike the convex-convex sun/planet gear contact, the planet/ring gear mesh has the convex-concave gear contact. This allows reduction of the effective gear facewidth in the planet/ring gear mesh in order to achieve a level similar to the contact stress. This typically makes the effective gear face width ratio  $K_b < 1.0$ ; or, typically, 0.7–0.9.

When the input torque and gear ratio are given, and the gear transmission density coefficient  $K_o$  is selected according to the application, volume functions allow for estimating both size and weight of the gearbox at a very preliminary stage of design for different options of gear arrangement.

### Volume Functions for Different Gear Arrangements

The volume functions are defined for four two-stage gear arrangements.

#### 1. External gear arrangement (Fig. 3)

The gearbox with this simple gear arrangement has a minimal number of gears and bearings; it is less expensive in production and

potentially more reliable. Its total gear ratio is:

$$u_t = u^I \times u^{II} \quad (19)$$

where indexes “I” and “II” are for the first and second stage, accordingly.

The volume function for this arrangement is:

$$F_v = \frac{u^I + 1}{n_b} \times \left( \frac{K_{v1}^I}{u^I} + n_b \times u^I \times K_{v2}^I \right) + \frac{u^I (u^{II} + 1)}{n_b} \times \left( \frac{n_b \times K_{v1}^{II}}{u^{II}} + u^{II} \times K_{v2}^{II} \right) \quad (20)$$

where:

$K_{v3}$	Volume utilization coefficient of the sun gear
$n_b$	Number of transmission branches

### 2. Epicyclic “star” arrangement (Fig. 4)

This gear arrangement provides a more compact and lighter gearbox in comparison with the external gear arrangement (Fig. 3), because the number of the transmission branches (planet gears) is typically three or more. The planet gears in this arrangement are not rotated around the sun gear, which makes their lubrication less complicated. The total gear ratio is:

$$u_t = p^I \times p^{II} \quad (21)$$

The volume function for this arrangement is:

$$F_v = F_{ve}^I + p^I \times F_{ve}^{II} \quad (22)$$

### 3. Epicyclic planetary arrangement (Fig. 5)

This epicyclic planetary gear arrangement is a more compact and lighter gearbox than the “star” example (Fig. 4), because the planet gears are installed on the carrier and involved in the planetary motion around the sun gear. The total gear ratio is:

$$F_v = F_{ve}^I + (1 + p^I) \times F_{ve}^{II} \quad (23)$$

The volume function for this arrangement is

$$u_t = (1 + p^I) \times (1 + p^{II}) \quad (24)$$

### 4. Epicyclic differential arrangement (Fig. 6)

The first stage is differential. The second stage has the “star” arrangement with the stationary carrier. Part of the transmitted power goes from the first-stage carrier directly to the output shaft. The remaining transmitted power goes from the first-stage ring gear to the second-stage sun gear and then through the planets to the second-stage ring gear—also connected to the output shaft.

This allows a reduction of the size and weight of the second-most loaded stage and makes the gearbox more compact and lighter than with the planetary arrangement (Fig. 5). The total gear ratio is:

$$u_t = 1 + p^I + p^I \times p^{II} \quad (25)$$

The volume function for this arrangement is:

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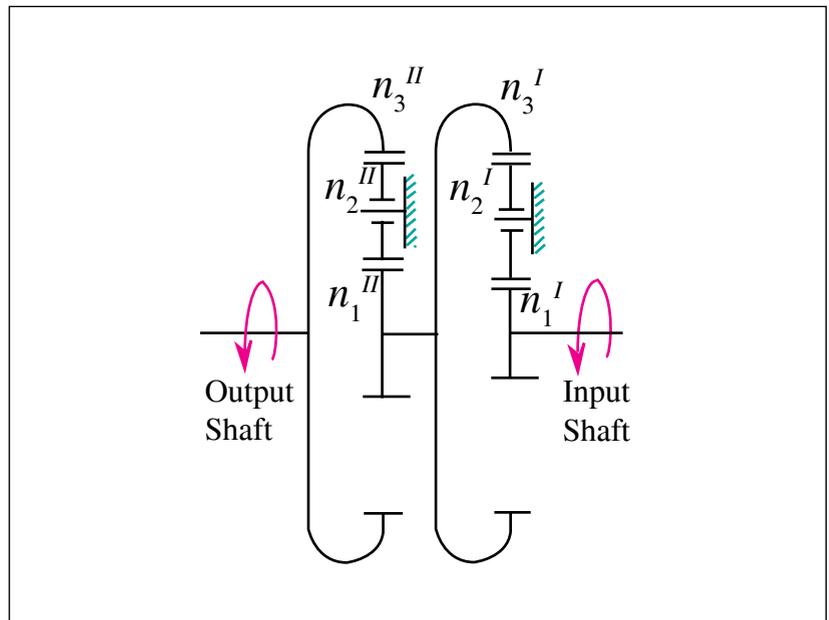


Figure 4—Two-stage epicyclic “star” gear train.

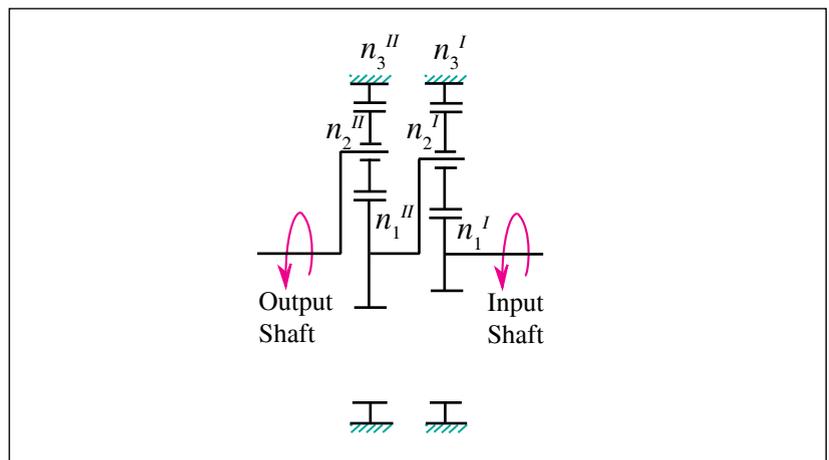


Figure 5—Two-stage epicyclic planetary gear train.

$$F_v = F_{ve}^I + \frac{p^I}{1+p^I} \times F_{ve}^{II} \quad (26)$$

### Internal Gear Ratio Optimization

Internal gear ratio distribution for multi-stage gearboxes can be optimized to achieve the minimum of the volume function. For two-stage gearboxes the minimum of the vol-

ume function  $F_v = f(u^I, u^{II})$  is achieved when the first derivatives  $d(F_v)/d(u^I)$  or  $d(F_v)/d(u^{II})$  equal zero.

Figure 7 presents a chart of the volume function versus the first-stage gear ratio for the two-stage external gear arrangement with one and two transmission branches with the total gear ratio of  $u = 15:1$ . The volume utilization coefficients are assumed for the pinions (driving gears)— $K_{v1} = 0.8$  and for the driven mating gears— $K_{v2} = 0.5$ .

Figure 7 makes clear that the minimum of the volume function for the two-transmission-branch arrangement is significantly lower than that for the one-transmission-branch arrangement because of load shearing; results of the volume function minimization and the optimal stage gear ratios are presented in Table 1.

Figure 8 shows the volume function versus the first-stage gear ratio charts for the two-stage epicyclic gear arrangements (Figs. 4–6) with the total gear ratio of  $u = 15:1$ ; both stages have three planets. The volume utilization coefficients are assumed for the sun gears to be— $K_{v1} = 0.8$ ; planet gears— $K_{v2} = 0.5$ ; and ring gears— $K_{v3} = 0.1$ . The effective gear face width ratio in the epicyclic gear stage is assumed as  $K_b = 0.75$ .

Figure 8 demonstrates that a minimum of the volume function for the epicyclic arrangement with the differential first stage has advantages in comparison with the “star” and planetary arrangements. This is because part of the power is transmitted from the first-stage carrier directly to the output shaft and the second stage is less loaded. The results of the volume function minimization and the optimal stage gear ratios are presented in Table 2.

After definition of the minimal total volume function and the stage gear ratios, the individual gear volume functions can be defined using Equations 10–11 and 16–18. The pitch cylinder volume of the individual gear, considering Equation 4, is then:

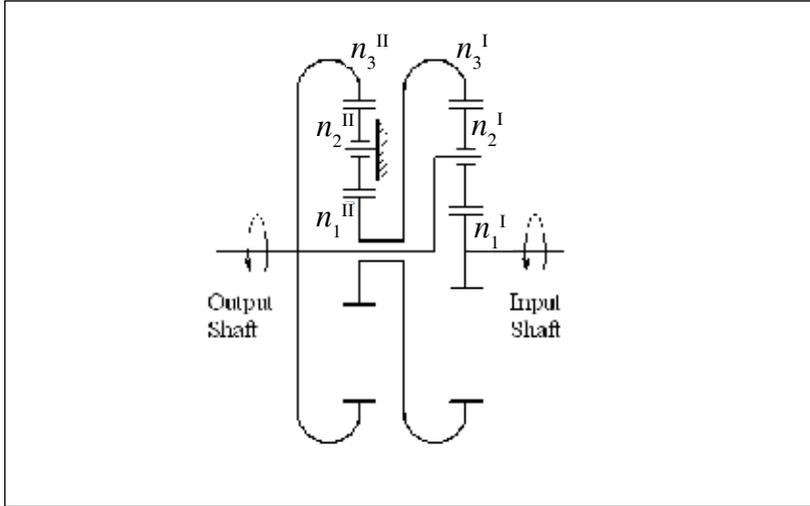


Figure 6—Two-stage epicyclic differential gear train.

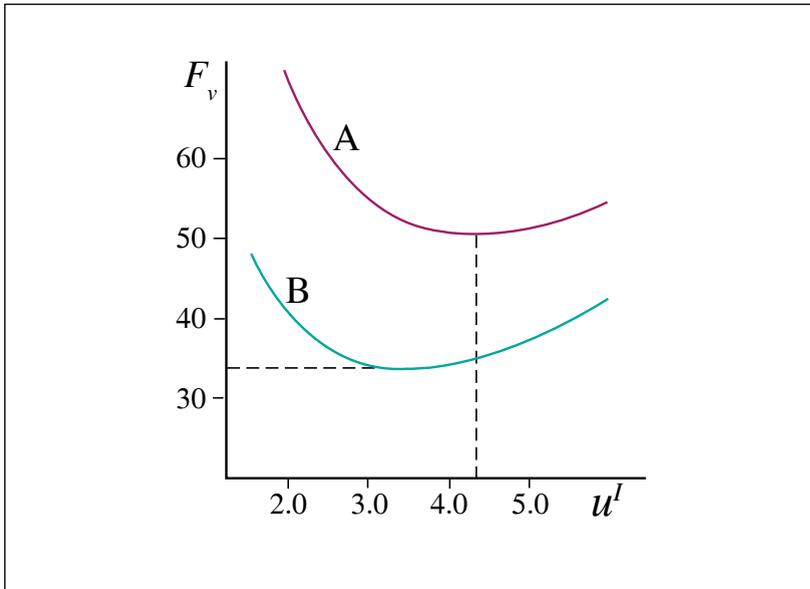


Figure 7—Two-stage external gear train volume function charts; A = with one-transmission branch (Fig. 3-B); B = with two transmission branches (Fig. 3-C).

Table 1—Results of the volume function minimization and the optimal stage gear ratios

Total gear ratio	15:1	
	1 (Fig. 3-B)	2 (Fig. 3-C)
Number of transmission branches	1 (Fig. 3-B)	2 (Fig. 3-C)
Minimum total volume function	50.469	33.697
1st stage volume function	12.476	8.303
2nd stage volume function	37.993	25.394
1st stage gear ratio	4.320:1	3.465:1
2nd stage gear ratio	3.472:1	4.329:1

$$V = \frac{\pi}{2} \times \frac{T_1}{K_0} \times F_v \quad (27)$$

From here:

$$d_w^2 \times b_w = \frac{4}{\pi} \times V = \frac{2 \times T_1}{K_v \times K_0} \times F_v \quad (28)$$

or:

$$d_w = \sqrt[3]{\frac{2 \times T_1 \times F_v}{K_v \times K_0 \times \psi}} \quad (29)$$

where:

- $K_v$  Volume utilization coefficient
- $\psi = b_w/d_w$  Aspect ratio that varies in a range of 0.05–1.2 or higher (Ref. 2)

This allows definition of all the gear diameters and size of all gears in assembly. However, the total volume and weight of the gearbox is not in direct proportion to its volume function. The share of the gear volume and weight is usually higher for simple arrangements like the external gear train. In a more complicated epicyclic gear arrangement, this share could be much lower because of a higher number and volume of other gearbox parts and components, such as carriers, bearings, shafts, lubrication system parts, etc. Statistical data of the gear volume share for the selected type of gear arrangement allow one to define the approximate size of the gearbox. In many cases, the gearbox is built in the overall mechanism assembly, and minimization of its size and weight should be considered to achieve optimum operating characteristics of the whole product, including, for example, cost, lifetime, noise and vibration.

The approach utilizing the volume functions allows estimation of the volume and weight of the gearbox for any multi-stage arrangement in a very early stage of prod-

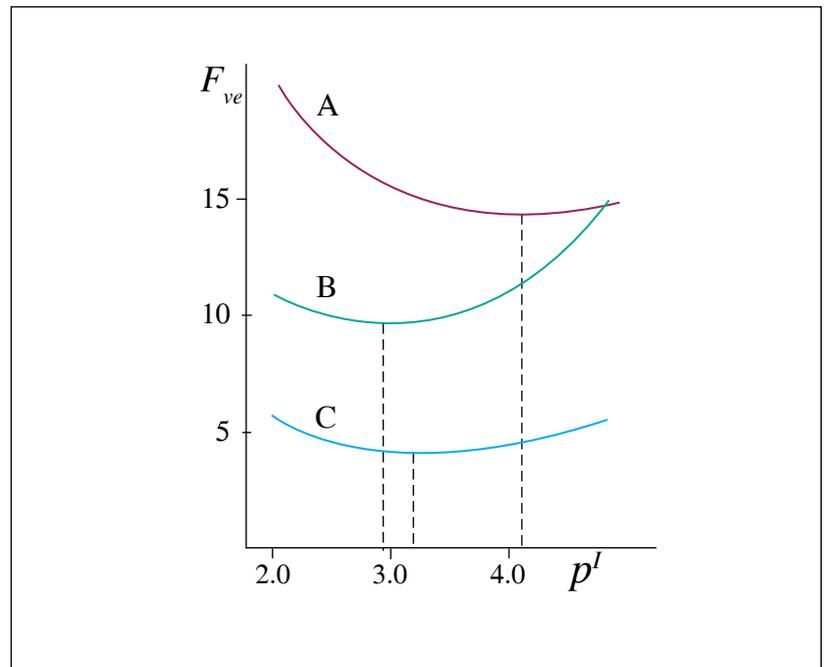
uct development. The next phase of gearbox design includes the gear and other component parameter calculation, and stress analysis produces a more accurate definition of the volume and weight of the gearbox.

### Gear Tooth Geometry for Higher Transmission Density

In most high-load-capacity gear transmissions, the tooth load on one flank is significantly higher and is applied for longer periods of time than for the opposite one; this creates the possibility of using gears with asymmetric teeth (Ref. 3).

The design intent of asymmetric gear teeth is to improve performance of the primary drive profiles at the expense of performance of the opposite coast profiles. The coast profiles are unloaded or lightly loaded during a relatively short work period. Asymmetric tooth pro-

**continued**



**Figure 8—Two-stage epicyclic gear train volume function charts; A = “star” arrangement (Fig. 4); B = planetary (Fig. 5); C = with first differential stage (Fig. 6).**

**Table 2—Results of the volume function minimization and the optimal stage gear ratios**

Total gear ratio	15:1		
	A	B	C
Gear arrangement ( Fig. 8)			
Min. total volume function	14.32	9.66	4.09
1st stage volume function	3.22	2.03	2.24
2nd stage volume function	11.09	7.63	1.85
1st stage planet/sun gear ratio	1.55:1	0.97:1	1.09:1
2nd stage planet/sun gear ratio	1.33:1	0.91:1	1.20:1
1st stage ring/sun gear ratio	4.11:1	2.93:1	3.19:1
2nd stage ring/sun gear ratio	3.65:1	2.82:1	3.40:1

files also make it possible to simultaneously increase the contact ratio and operating pressure angle beyond conventional gears' limits. The main advantage of asymmetric gears is reduction of gear dimensions while maintaining the allowable contact stress level, resulting in higher transmission density coefficient  $K_o$ . Asymmetric gear testing at the rotorcraft division of the Boeing Company demonstrated superior scoring performance when compared to conventional (baseline) symmetric gears. The mean value for a limited data set showed an improvement of approximately 25% (Ref. 4).

The combination of the volume function approach and gears with asymmetric teeth achieved extremely high power density in the turboprop engine gearbox (Ref. 5).

### Summary

- The dimensionless volume functions were introduced and correlated to gear transmission density.
- The gear train volume functions are described and defined for the gear pair and epicyclic gear stage.
- Different two-stage gear train arrangement volume functions were analyzed to find their minimum and optimal stage gear ratio distribution.

- Application of gears with asymmetric teeth provides enhanced gear drive transmission density. 

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