When designing gears, the engineer is often faced with the problem of selecting the number of teeth in each gear, so that the gear train will provide a given speed ratio. This article will describe a microcomputer program that determines such combinations of gear teeth quickly, easily, and accurately.

The program is founded upon the theorem that a denumerable infinity of rational numbers can be found between any two real numbers whose difference is greater than zero. Practically, if the acceptable error for the desired gear ratio is of the order of several hundred times the minimum numerical values that the computer can process without truncation errors, it is usually possible to find multiple combinations of gear teeth which provide that ratio to within required accuracy.

Program Description

Input to the method are the desired ratio R and the acceptable error E.[1] The method produces two integers P and Q, where P/Q differs from R by an amount equal to or less than E. The search for P and Q is begun by defining quantities a_1, y_1, p_1, and q_1 as

$$a_1 = \text{INT}(R) \qquad y_1 = \text{FRC}(R)$$
$$p_1 = a_1 \qquad q_1 = 1 \tag{1}$$

Finding Gear Teeth Ratios

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Quantities a_2, y_2 p_2, and q_2 and qa_2 are then defined according to

$$a_2 = \text{INT}\left(\frac{1}{y_1}\right) \qquad y_2 = 1 - a_2 y_1$$
$$p_2 = a_1 a_2 + 1 \qquad q_2 = a_2 \tag{2}$$

If $y_2 = 0$ the process stops because $P = p_2$ *and* $Q = q_2$. If $y_2 \neq$o the process continues until

$$E \geqq \left| R - \frac{p_N}{q_N} \right| \tag{3}$$

where p_N and q_N represent the values of p_n and $q_n$which satisfy (3) for the smallest n. In (3), p_n and q_n are determined by the relations

$$a_n = \text{INT}\left(\frac{y_{n-2}}{y_{n-1}}\right) \qquad y_n = 1 - a_n y_{n-1} + y_{n-2}$$
$$p_n = a_n p_{n-1} + p_{n-2} \qquad q_n = a_n q_{n-1} + q_{n-2} \tag{4}$$

for $n \geqq 3$. The computational procedure using these relations is described in the flowchart in Fig. 1.

Fig. 2 is a flowchart of the main program, Gearratio. It

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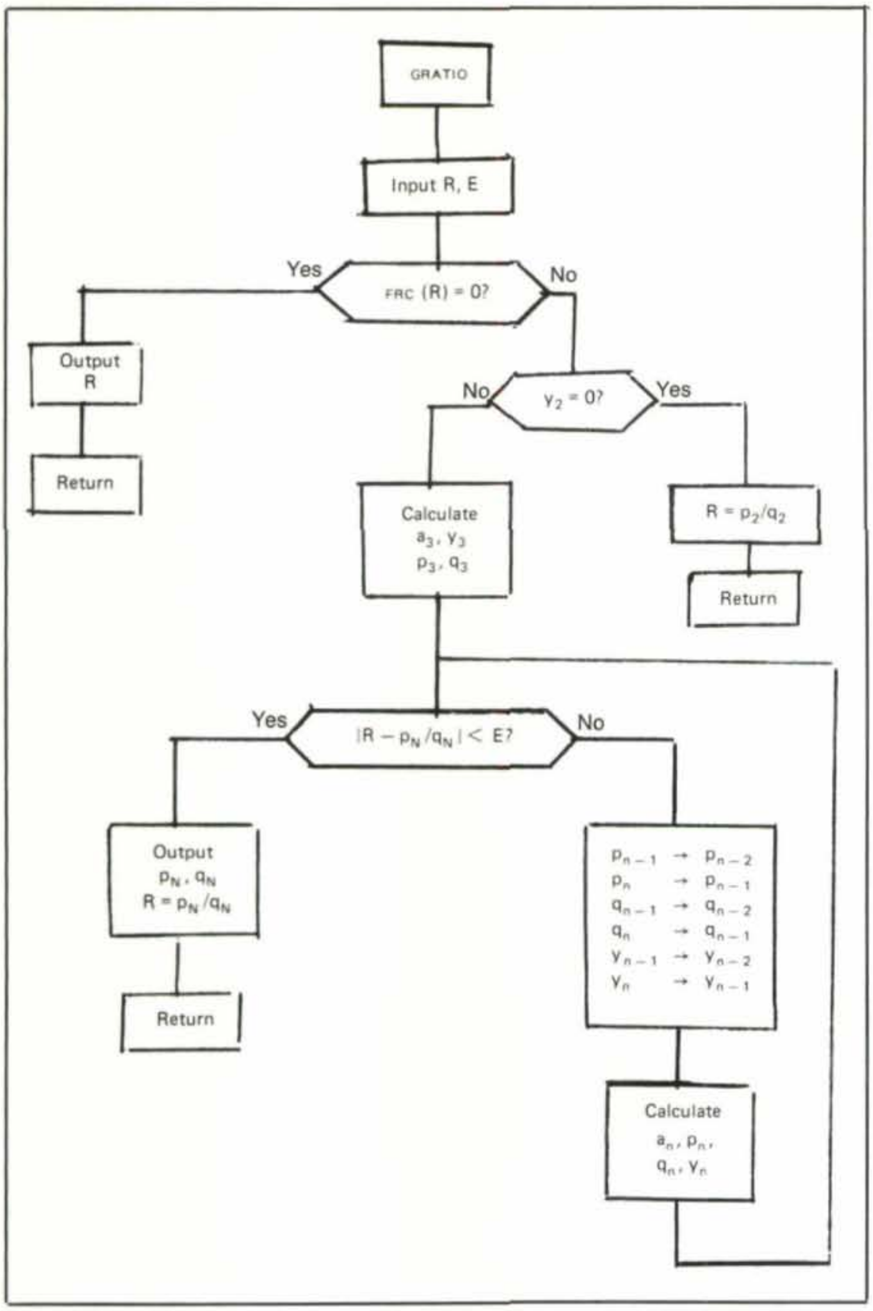

Fig. 1—Flowchart for the subprogram GRATIO.

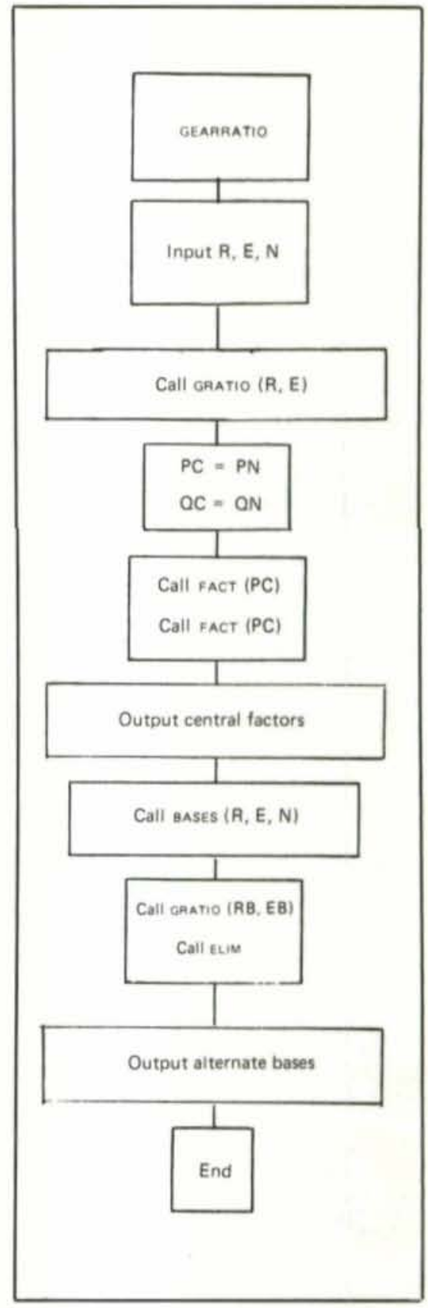

Fig. 2—Flowchart for the main program Gearratio.

Nomenclature

a_n	value of parameter a at the nth step	p_N	final value of p_n
E	acceptable error in approximating the desired gear ratio	Q	denominator of a rational number approximating R
FRC(R)	fractional part of R: FRC(R) = R − INT(R)	q_n	value of Q at the nth step
INT(R)	largest integer in R	q_N	final value of q_n
P	numerator of a rational number (fraction) approximating R	R	desired gear ratio
p_n	value of P at the nth step	R_i	additional gear ratios which differ from R by less than E
		y_n	value of y at the nth step

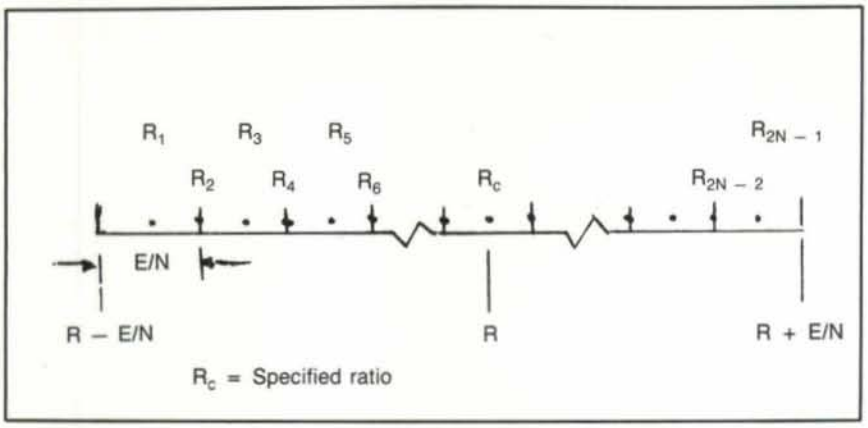

Fig. 3—Locations of R_i and R in the interval E. N subdivisions of length E/N are shown.

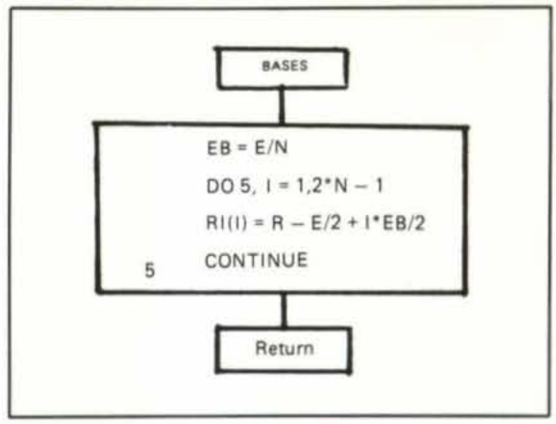

Fig. 4—Flowchart for subroutine BASES.

uses the acceptable error E to define an interval $R - E/2$ to $R + E/2$. Other ratios R_i can be selected using error limits that are reduced in proportion to the number of subdivisions of E (Fig. 3). Division of E into N equal subdivisions each of length E/N is performed by the four-line subprogram labeled BASES (Fig. 4) This subroutine then locates $2N - 1$ additional base ratios R_i by the relation

$$R_i = R - \frac{E}{2} + (2i - 1)\,\frac{E}{2N} \qquad (5)$$

Each base ratio is associated with a permissible error of magnitude E/N, to ensure that all calculated values fall within the originally specified interval. Values P and Q for each additional base ratio are calculated after the original base ratio R is used to find the central P and Q values to within permissible error E.

Different P and Q values can be found by a variety of methods. For example, the permissible error E can be allowed to grow as the base ratios move closer to the center of the range. Or, one can use an uneven distribution of base ratios over the interval and locate the base ratio at other than the center of the subintervals of length E/N. In the following examples, the results are generated using only base ratios at a set of equally spaced points associated with an error of $E/100$. This division produces 198 additional base ratios (199 minus the central base ratio already calculated), including duplicates. Elimination of duplicates is achieved by the subroutine ELIM (Fig. 5).

After all duplicates are eliminated the remaining values of P and Q are decomposed into their prime factors by the subroutine FACT (Fig. 6). Output of the factors is controlled by a write statement in this subroutine.

Example 1

In this example problem we will find the numbers of teeth in each gear in a gear train which is to have an overall speed ratio of 2.94643 ± 0.0001. There should be at least 18 and at most 200 teeth on each gear.

Because P and Q generally do not contain an equal number of factors, it may be necessary to append unit factors to each. The factors in P and Q are then arranged in ascending or descending order and grouped in pairs. Using the gear train shown schematically in Fig. 7, P and Q can be written in the form

$$\frac{N_1}{N_2}\;\frac{N_3}{N_4}\;\frac{N_5}{N_6}$$

Comparison with gear train shows that (6) implies a gear train of six gears in which N_1 and N_2, N_3 and N_4, and N_5 and N_6 are in mesh. Gear N_1 is on shaft 1, gears N_2 and N_3 are on shaft 2, gears N_4 and N_5 are on shaft 3, and gear N_6 is on shaft 4.

After each P/Q ratio is arranged as in (6), any pair which contains a factor less than 18 is multiplied by the smallest integer required to make that factor equal to or greater than 18. When this is completed, it may be necessary to rearrange the P and Q factors to have the modified factors again be in ascending or descending order. Had the ratio in this example been unusually large, it is possible that this multiplication could cause the other factor in a P/Q pair to be larger

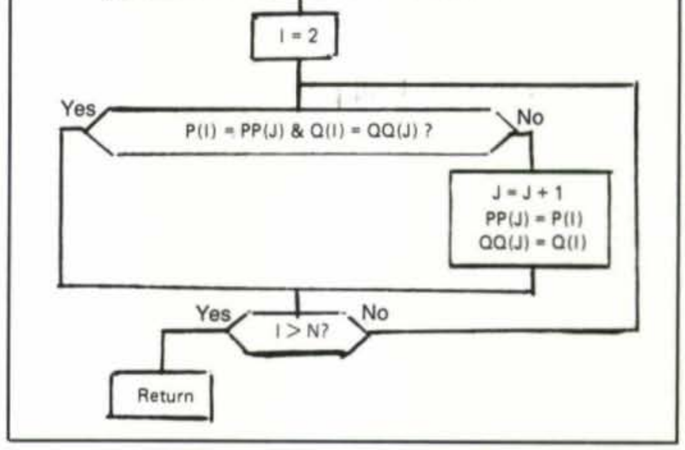

Fig. 5—Flowchart for subprogram ELIM.

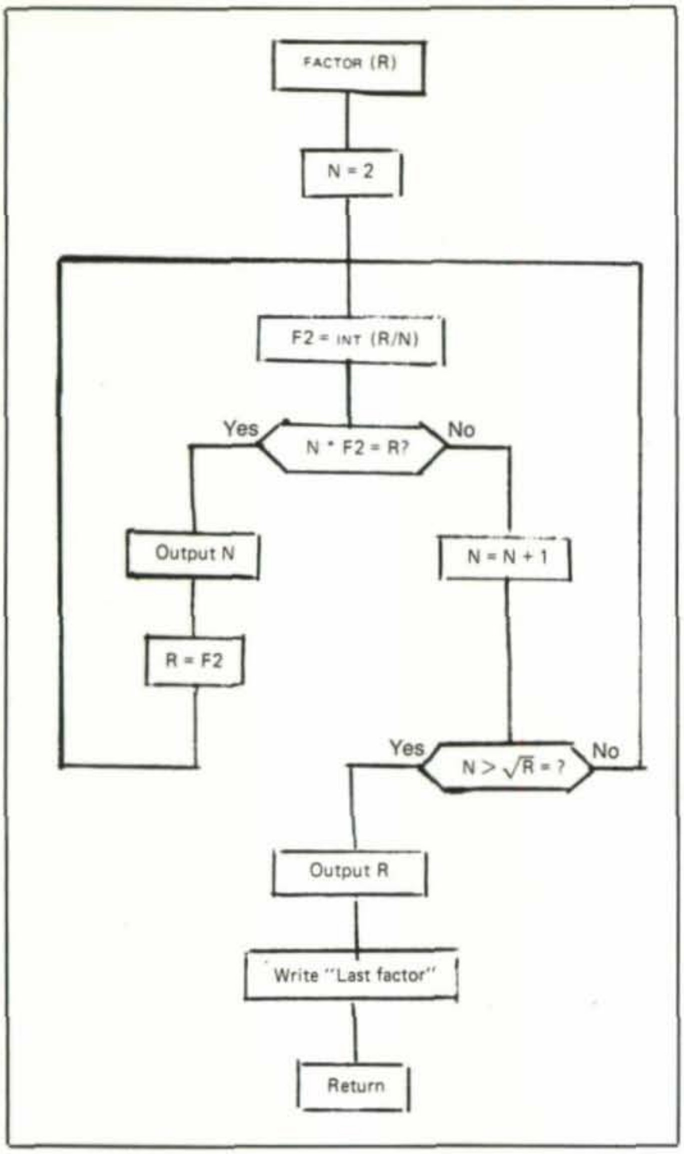

Fig. 6 – Flowchart for subprogram FACT.

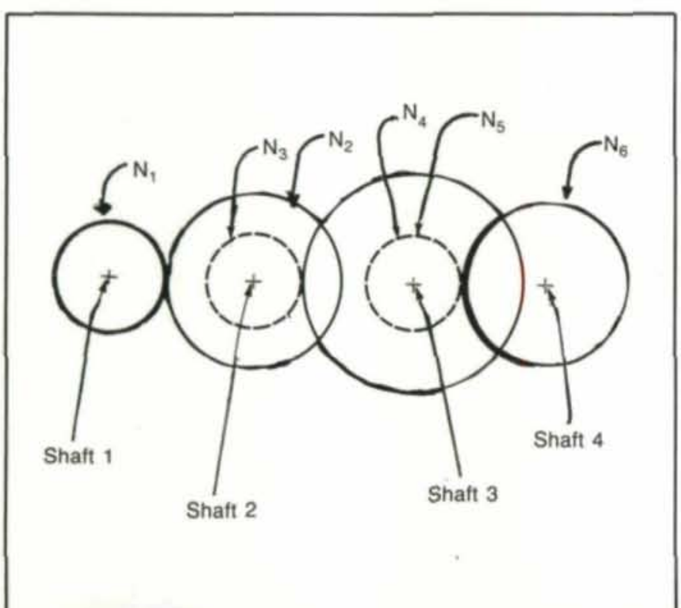

Fig. 7 – Schematic of gear train.

Table I

Tooth Combinations to Provide a Gear Ratio Of 2.94643 ± 0.0001 (18 ⩽ N ⩽ 200)

Ratio Number	Program Output Ratio	Gear Tooth Numbers	Error (x 10^6)
1	$\frac{3(5)11}{(2^3)(7)}$	$\frac{45}{24}\ \frac{33}{21}$	1
2	$\frac{2^4(79)}{3(11)13}$	$\frac{32}{22}\ \frac{79}{39}$	43
3	$\frac{29(127)}{2(5^4)}$	$\frac{29}{25}\ \frac{127}{50}$	30
4	$\frac{2(7^2)23}{(3^2)5(17)}$	$\frac{28}{20}\ \frac{28}{18}\ \frac{46}{34}$	25
5	$\frac{2(31)47}{23(43)}$	$\frac{47}{23}\ \frac{62}{43}$	19
6	$\frac{(2^6)\ 61}{(5^2)\ 53}$	$\frac{64}{53}\ \frac{61}{25}$	15
7	$\frac{3(5)11}{23(7)}$	$\frac{33}{23}\ \frac{20}{28}$	14
8	$\frac{137(163)}{11(13)53}$	$\frac{163}{143}\ \frac{137}{53}$	1
9	$\frac{(2^2)19(131)}{31(109)}$	$\frac{76}{31}\ \frac{131}{109}$	4
10	$\frac{(2^2)7(167)}{3\ (23^2)}$	$\frac{28}{23}\ \frac{167}{69}$	10
11	$\frac{2(41)53}{(5^2)\ 59}$	$\frac{53}{25}\ \frac{82}{59}$	11
12	$\frac{37(113)}{3(11)43}$	$\frac{37}{33}\ \frac{113}{43}$	11
13	$\frac{2(19)97}{(3^2)\ 139}$	$\frac{194}{139}\ \frac{38}{18}$	13
14	$\frac{2(17)89}{13(79)}$	$\frac{68}{26}\ \frac{89}{79}$	16
15	$\frac{2(13^2)7}{11(73)}$	$\frac{28}{22}\ \frac{169}{73}$	21
16	$\frac{31(71)}{(3^2)\ 83}$	$\frac{62}{18}\ \frac{71}{83}$	22
17	$\frac{17(191)}{2(19)29}$	$\frac{34}{38}\ \frac{191}{58}$	31
18	$\frac{(3^4)\ 53}{31(47)}$	$\frac{81}{47}\ \frac{53}{31}$	35
19	$\frac{7(173)}{3(137)}$	$\frac{42}{18}\ \frac{173}{137}$	42
20	$\frac{(2^2)\ 3(17^2)}{11(107)}$	$\frac{24}{22}\ \frac{289}{107}$	44

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Table II
Tooth Combinations to Provide a Gear Ratio Of 5.17220 ± 0.0001 (18 ⩽ N ⩽ 200)

Ratio Number	Program Output Ratio	Tooth Combinations	Error (x 10^6)
1	$\frac{(11)(71)}{151}$	$\frac{71}{18}\frac{198}{151}$	15
2	$\frac{(7)(41)(47)}{(2^4)(163)}$	$\frac{47}{18}\frac{63}{24}\frac{123}{163}$	37
3	$\frac{29(173)}{2(5)97}$	$\frac{58}{20}\frac{173}{97}$	35
4	$\frac{5(7)103}{17(41)}$	$\frac{70}{34}\frac{103}{41}$	34
5	$\frac{3(17)43}{(2^3)(53)}$	$\frac{43}{18}\frac{51}{24}\frac{54}{53}$	30
6	$\frac{(2^2)(3^2)5(131)}{47(97)}$	$\frac{180}{97}\frac{131}{47}$	13
7	$\frac{(2^7)(3)37}{41(67)}$	$\frac{96}{41}\frac{148}{67}$	12
8	$\frac{3(4)7(113)}{2(31)37}$	$\frac{105}{37}\frac{113}{62}$	12
9	$\frac{(2^3)(5)199}{(3^4)(19)}$	$\frac{40}{19}\frac{199}{81}$	10
10	$\frac{41(137)}{2(3)181}$	$\frac{123}{18}\frac{137}{181}$	9
11	$\frac{(2^2)(3)13(31)}{5(11)17}$	$\frac{48}{20}\frac{26}{22}\frac{62}{34}$	8
12	$\frac{2(17)197}{5(7)37}$	$\frac{36}{18}\frac{34}{20}\frac{197}{37}$	0
13	$\frac{61(97)}{(2^3)(11)(13)}$	$\frac{61}{26}\frac{97}{44}$	3
14	$\frac{6(17)53}{16(37)}$	$\frac{106}{26}\frac{85}{67}$	16
15	$\frac{2(41)89}{17(83)}$	$\frac{82}{34}\frac{178}{83}$	18
16	$\frac{(7^2)(19)}{(2^2)(3^2)5}$	$\frac{76}{36}\frac{49}{20}$	21
17	$\frac{(2^2)(5)11(43)}{31(59)}$	$\frac{55}{31}\frac{172}{59}$	25
18	$\frac{2(29)131}{13(113)}$	$\frac{116}{26}\frac{131}{113}$	26
19	$\frac{2(13)149}{7(107)}$	$\frac{36}{18}\frac{39}{21}\frac{149}{107}$	30
20	$\frac{(2^6)(3^2)11}{(5^2)(7^2)}$	$\frac{64}{25}\frac{99}{49}$	44
21	$\frac{23(47)}{11(19)}$	$\frac{46}{19}\frac{47}{22}$	49

than 200. This P/Q would then have to be treated by the same method.

Twenty-one tooth ratios are left after duplicate gears and those with more than 200 teeth are eliminated. These are displayed in the second column of Table I in the form returned to the main program by FACT. Column three is the rewritten form of these tooth ratios after multiplication factors have been applied, so that no gear has fewer than 18 teeth and after the P/Q factors have been rearranged into ascending or descending order. According to the form described by (6), it follows that gear ratio 1 in Table I represents a shorter gear train. It has only three shafts. Gear 1 has 45 teeth and is on shaft 1, the input shaft. It is in mesh with gear 2, with 24 teeth, mounted on shaft 2. Shaft 2 also holds gear 3, which has 33 teeth and is in mesh with gear 4. Gear 4 has 21 teeth and is mounted on shaft 3, the output shaft.

Example 2

We will repeat the selection process used in Example 1 for the ratio 5.17220 ± 0.0001. E will be divided into 100 and 113 intervals and the same criteria for the acceptable number of teeth on each gear will be used.

The results for 100 subintervals are displayed in Table II. Division into 113 subintervals provided only two more ratios, as shown in Table III. As these two examples demonstrate, the number of ratios obtained depends on both the input ratio and on the number of subdivisions of the permissible error E.

Note that the maximum error in all three tables is 49×10^{-6}.

Table III
Additional Tooth Combinations Found for R = 5.17220 Using N = 113 (18 ⩽ N ⩽ 200)

Ratio Number	Program Output Ratio	Tooth Combinations	Error (x 10^6)
1	$\frac{(2^5)5(61)}{3(17)37}$	$\frac{64}{34}\frac{30}{18}\frac{61}{37}$	31
2	$\frac{19(2^3)\ 67}{11(179)}$	$\frac{76}{22}\frac{36}{18}\frac{134}{179}$	31

References

1. ORTHWEIN, W.C., "Determination of Gear Ratios," ASMEJ. of Mechanical Design, Vol. 104, 1982, pp. 775-777.

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